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Vorlesung Differentialgeometrie

From Spec$\mathbb{R}[X]$ to the hyperbolic plane $\mathbb{H}$.

In this note we explain first the equation:

$$\mathrm{Spec}(\mathbb{R}[X]) = \mathbb{R} \cup \mathbb{H}$$

which reads as:

The spectrum of the ring $\mathbb{R}[X]$ of polynomials with real coefficients and indeterminate X is the union of the real line $\mathbb{R}$ and the hyperbolic plane $\mathbb{H}$.

The spectrum of the ring $\mathbb{R}[X]$ is a set having as elements the prime ideals of the ring $\mathbb{R}[X]$. The ring $\mathbb{R}[X]$ has two kinds of prime ideals: For $r \in \mathbb{R}$ the ideal $I_r := (X - r) \subset \mathbb{R}[X]$ is a prime ideal of the first kind. For $b, c \in \mathbb{R}$ with $b^2 - 4c < 0$ the ideal $I_{b,c} := (X^2 + bX + c) \subset \mathbb{R}[X]$ is a prime ideal of the second kind. Observe, that the ideals $I_r, I_{b,c}$ are indeed maximal ideals in $\mathbb{R}[X]$ and that the quotient fields are isomorphic to the field $\mathbb{R}$ for prime ideals I_r of the first kind and to the field $\mathbb{C}$ for prime ideals of the second kind.

Let $\mathbb{H}$ be the set $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ of complex numbers $z = x + yi$, $x, y \in \mathbb{R}$, with strictly positive imaginary part $\mathrm{Im}(z) := y$. A polynomial $X^2 + bX + c$, $b, c \in \mathbb{R}$, $b^2 - 4c < 0$, has exactly one root $z \in \mathbb{H}$. We have $b = 2\mathrm{Re}(z)$ and $b^2 - 4c = -\mathrm{Im}(z)^2$. We denote the ideal $I_{b,c}$ also by I_z, where $z \in \mathbb{H}$ is a root of $X^2 + bX + c$. For $z \in \mathbb{H}$ we denote by $b(z) = 2\mathrm{Re}(z)$ and $c(z) = \mathrm{Im}(z)^2 + \mathrm{Re}(z)^2$.

Putting notations together, we have as sets:

$$\mathrm{Spec}(\mathbb{R}[X]) = \{I_r \mid r \in \mathbb{R}\} \cup \{I_z \mid z \in \mathbb{H}\} = \mathbb{R} \cup \mathbb{H}$$

where we have used the identifications $r \in \mathbb{R} \mapsto I_r \in \mathrm{Spec}(\mathbb{R}[X])$ and $z \in \mathbb{H} \mapsto I_z \in \mathrm{Spec}(\mathbb{R}[X])$.

We wish to do more and define a metric on the set $\mathbb{H}$, that we consider as the set of prime ideals of second kind in $\mathbb{R}[X]$. Let $p, q \in \mathbb{H}, p \neq q$. In the vector space $\mathbb{R}[X]$ we consider the parametrized line

$$t \in \mathbb{R} \mapsto w_t(X) := (1-t)(X^2 + b(p)X + c(p)) + t(X^2 + b(q)X + c(q))$$

$$= X^2 + (b(p) + t(b(q) - b(p)))X + c(p) + t(c(q) - c(p)) \in \mathbb{R}[X]$$

The ideal $(w_t(X))$ is a prime ideal of the second kind for all $t \in \mathbb{R}$ with negative discriminant $\Delta_{p,q}(t) := (b(p) + t(b(q) - b(p)))^2 - 4(c(p) + t(c(q) - c(p)))$. The polynomial function

$$\Delta_{p,q} : t \in \mathbb{R} \mapsto \Delta_{p,q}(t) \in \mathbb{R}$$

has leading term $t^2(b(q)-b(p))^2$. We have two cases:

Case 1. $b(p) \neq b(q)$. We may assume $b(p) = 2\mathrm{Re}(p) < 2\mathrm{Re}(q) = b(q)$. The coefficient of t^2 in $\Delta_{p,q}$ is strictly positive and $\Delta_{p,q}(0)$ as $\Delta_{p,q}(1)$ are negative. Hence the equation $\Delta_{p,q}(t) = 0$ has a root $t^-_{p,q} < 0$ and a root $t^+_{p,q} > 1$.

For $t \in]t^-_{p,q}, t^+_{p,q}[$ let $z_{p,q}(t) \in \mathbb{H}$ be the root of $w_t(X)$. Hence,

$$t \in]t^-_{p,q}, t^+_{p,q}[\mapsto z_{p,q}(t) \in \mathbb{H}$$

is a path in $\mathbb{H}$ with $p = z_{p,q}(0)$ and $q = z_{p,q}(1)$.

An important observation is the following:

Lemma 1 *The image of the path $t \in]t^-_{p,q}, t^+_{p,q}[\mapsto z_{p,q}(t)$ is the trace on $\mathbb{H}$ of the circle in $\mathbb{C}$ that passes through p and q and with center on $\mathbb{R}$.*

Aufgabe 1. Beweis!

We define the number

$$\mathrm{Dist}(p,q) := \frac{1}{2}|\log \frac{(1-t^-_{p,q})(t^+_{p,q}-0)}{(0-t^-_{p,q})(t^+_{p,q}-1)}|$$

Observe:

$$\log \frac{(1-t^-_{p,q})(t^+_{p,q}-0)}{(0-t^-_{p,q})(t^+_{p,q}-1)} = -\log \frac{(1-t^-_{q,p})(t^+_{q,p}-0)}{(0-t^-_{q,p})(t^+_{q,p}-1)}$$

so in order to make $\mathrm{Dist}(p,q)$ symmetric in p and q we use the absolut value.

Case 2. $b(p) = b(q)$. We may assume $c(p) < c(q)$. We consider again the line $w_t(X)$. Now $\Delta_{p,q}(t) = b(p)^2 - 4(c(p) + t(c(q) - c(p)))$. We have again $\Delta_{p,q}(t) < 0$ for $t \in]t^-_{p,q}, t^+_{p,q}[$ with

$$t^-_{p,q} = \frac{b(p)^2 - 4c(p)}{4(c(q)-c(p))} < 0, t^+_{p,q} = +\infty$$

We consider again the path $t \in]t^-_{p,q}, t^+_{p,q}[\mapsto z_{p,q}(t) \in \mathbb{H}$ of roots of $w_{p,q}(t)$ and have:

Lemma 2 *The image of the path $t \in]t^-_{p,q}, t^+_{p,q}[\mapsto z_{p,q}(t)$ is the trace on $\mathbb{H}$ of the real line in $\mathbb{C}$ that passes through p and q.*

Aufgabe 2. Beweis!

We define the number:

$$\mathrm{Dist}(p,q) := \lim_{s \to +\infty} \frac{1}{2}|\log \frac{(1-t^-_{p,q})(s-0)}{(0-t^-_{p,q})(s-1)}|$$

We wish to prove, that this expression is a distance function on $\mathbb{H}$. But we go in small steps, in infinitesimal steps.

Lemma 3 *For $p \in \mathbb{H}$ and $h \in \mathbb{C}$ we have:*

$$\lim_{s \to 0, s>0} \frac{1}{s} \mathrm{Dist}(p, p+sh) = \frac{|h|}{\mathrm{Im}(p)}$$

Aufgabe 3. Beweis!

For $p \in \mathbb{H}$ fixed, we consider the function $\mathrm{Dist}_p : q \in \mathbb{H} \mapsto \mathrm{Dist}(p,q)^2 \in \mathbb{R}$.

Lemma 4 *The functions $\mathrm{Dist}_p : \mathbb{H} \to \mathbb{R}$ are twice differentiable. The function Dist_p is a Morse function with exactly one critical point, which is at p. The Hessian B_p at p is positive definite. The map $p \in \mathbb{H} \mapsto B_p$ is a Riemannian metric on $\mathbb{H}$. Moreover, for $p \in \mathbb{H}, u, v \in \mathbb{C}$ we have:*

$$B_p(u,v) = \frac{\mathrm{Re}(u\bar{v})}{\mathrm{Im}(p)^2}$$

Aufgabe 4. Beweis!

Definition. The Riemannian manifold $(\mathbb{H}, B)$ is called the hyperbolic plane $\mathbb{H}$.

We continue studying the hyperbolic plane. The parametrized curves $t \in]t^-_{p,q}, t^+_{p,q}[\mapsto z_{p,q}(t) \in \mathbb{H}, p, q \in \mathbb{H}, p \neq q$, are not geodesics. For instance, take $p = i, q = 2i$, then the parametrized curve $t \in]\frac{-1}{3}, +\infty[\mapsto z_{p,q}(t) = i + 3ti \in \mathbb{H}$ is not a geodesic. The curve $s \in \mathbb{R} \mapsto e^s i \in \mathbb{H}$ has the same image and is a parametrized geodesic.

Aufgabe 5. Die Kurve $s \in \mathbb{R} \mapsto e^s i \in \mathbb{H}$ ist eine Geodätische. Hinweis: Die Abbildung $x + yi \in \mathbb{H} \mapsto -x + yi \in \mathbb{H}$ ist eine Isometrie von $\mathbb{H}$. Die Parametrisierung ist bogenlängentreu.

A real two by two matrix $\begin{matrix} a & b \\ c & d \end{matrix}$ of determinant 1 defines an isometry $z \in \mathbb{H} \mapsto \frac{az+b}{cz+d}$ of $\mathbb{H}$. The acion $z \in \mathbb{H} \mapsto \frac{az+b}{cz+d}$ is called the fractional action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}$.

Aufgabe 6. Beweis!

The curve $t \in]t^-_{p,q}, t^+_{p,q}[\mapsto z_{p,q}(t) \in \mathbb{H}$ is determined by its end points $p' := z_{p,q}(t^-_{p,q})$ and $q' := z_{p,q}(t^+_{p,q})$. We may assume $p' \neq \infty$. We have $p' \neq q'$. Define $a := \frac{q'}{q'-p'}, b := p', c := \frac{1}{q'-p'}, d := 1$. Here we use: $\frac{\infty}{\infty} = 1$. The isometry of $\mathbb{H}$ defined by the real matrix $\begin{matrix} a & b \\ c & d \end{matrix}$ of determinant 1 maps the geodesic through $i, 2i$ upon the curve $t \in]t^-_{p,q}, t^+_{p,q}[\mapsto z_{p,q}(t) \in \mathbb{H}$, hence $s \in \mathbb{R} \mapsto (ae^s i + b)/(ce^s i + d) \in \mathbb{H}$ is a geodesic reparametrisation of the curve $t \in]t^-_{p,q}, t^+_{p,q}[\mapsto z_{p,q}(t) \in \mathbb{H}$.

We now wish to show that $(p,q) \in \mathbb{H} \times \mathbb{H} \mapsto \mathrm{Dist}(p,q) \in \mathbb{R}$ is a distance on $\mathbb{H}$. We proceed in several steps.

Step 1. For $p, q \in \mathbb{H}$ let PDist : $\mathbb{H} \times \mathbb{H} \to \mathbb{R}$ denote the path distance deduced from the Riemannian metric B, i.e.

$$\mathrm{PDist}(p,q) := \mathrm{Infimum}_\gamma \int_0^1 \sqrt{B_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))} dt$$

Here γ runs over the set of continuous differentiable curves $\gamma : [0,1] \to \mathbb{H}$ with $\gamma(0) = p, \gamma(1) = q$.

Step 2. First, we assume that $p = Ai \in \mathbb{H}$ and $q = Bi, A, B \in \mathbb{R}, A, B > 0$. In this case we observe, that the curve $z_{p,q} : [0,1] \to \mathbb{H}$ realizes the $\mathrm{PDist}(p,q)$. To see this, use the linear projection map $\pi : \mathbb{H} \to \mathbb{H}$ given by $\pi(x + yi) := yi, x \in \mathbb{R}, y \in \mathbb{R}, y > 0$. For any path γ we have the inequality:

$$\int_0^1 \sqrt{B_{\pi(\gamma(t))}(\pi(\dot\gamma(t)), \pi(\dot\gamma(t)))} dt \leq \int_0^1 \sqrt{B_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))} dt$$

since for all $p \in \mathbb{H}$ all $u \in \mathbb{C}$ we have $B_{\pi(p)}(\pi(u), \pi(u)) \leq B_p(u,u)$.

Step 3. For all $p, q \in \mathbb{H}, p \neq q$ the curve $z_{p,q} : [0,1] \to \mathbb{H}$ realizes the infimum in the definition of $\mathrm{PDist}(p,q)$. Use an appropriate two by two matrix $\begin{matrix} a & b \\ c & d \end{matrix}$ that maps by the fractional action the points p, q to points P, Q with $\mathrm{Re}(P) = \mathrm{Re}(Q) = 0$. This action maps the curve $z_{p,q} : [0,1] \to \mathbb{H}$ to the curve $z_{P,Q} : [0,1] \to \mathbb{H}$. Since the fractional action is by isometries, we conclude with Step 2.

Step 4. We have

$$\mathrm{PDist}(p,q) = \int_0^1 \sqrt{B_{z_{p,q}(t)}(\dot z_{p,q}(t), \dot z_{p,q}(t))} dt = \mathrm{Dist}(p,q)$$

We conclude, that $\mathrm{Dist}(p,q)$ defines a metric: $\mathrm{Dist}(p,q)$ is by step 4 the path metric $\mathrm{PDist}(p,q)$, hence satisfies the triangular inequality. $\mathrm{Dist}(p,q) = 0$ only if $p = q$. Also we have the symmetry $\mathrm{Dist}(p,q) = \mathrm{Dist}(q,p)$.