Tête-à-tête twists and geometric monodromy. **Unstable preprint version**.

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Introduction. Let (Σ, Γ) be a pair consisting of a compact, connected, oriented, surface Σ with non empty boundary $\partial\Sigma$ and a finite graph Γ that is embedded in the interior $\Sigma \setminus \partial\Sigma$ of Σ. We assume that the surface Σ is a regular neighborhood of the graph Γ and that the embedded graph has the pur or mexed tête-à-tête property, which will be defined later in this paper. Moreover, for each pair (Σ, Γ) with the tête-à-tête property we will construct a relative mapping class T_Γ on $(\Sigma, \partial\Sigma)$. We call the mapping classes resulting from this construction tête-à-tête twists.

A surface of genus g and with r boundary components carries up to congruence by homeomorphism of the surface only finitely many graphs with the tête-à-tête property and hence for fixed (g, r) there are up to conjugacy only finite by many mapping classes, which are tête-à-tête twists.

The main theorem of this paper asserts:

Theorem 1. *The geometric monodromy diffeomorphism of an isolated plane curve singularity is a tête-à-tête twist.*

As a corollary, we obtain a very strong topological restriction for mapping classes, that are geometric monodromies of isolated plane curve singularities.

Conversely, a tête-à-tête twist can in general not be realized as geometric monodromy of a plane curve singularity, but tête-à-tête twists and geometric monodromies of plane curve singularities have many properties in common. For instance, as for geometric monodromies of plane curve singularities, a suitable high power of a tête-à-tête twist is a multi Dehn twist. In fact we show, see Theorem 3, that under iteration the growth of the length of curves on the surface is at most linear in the iteration exponent.

Sebastian Baader has suggested that every tête-à-tête twist appears as the monodromy diffeomorphism of a link in the 3-sphere and he asked for a knot theoretic proof that adds and removes (positiv?) Hopf bands. I thank him for this suggestion and question. We will define later for graphs with tête-à-tête property the laterna property.

Theorem 2. *Every tête-à-tête twist of a graph with the laterna property appears as the monodromy diffeomorphism of a classical fibred link.*

We propose to call the corresponding knots and links *tête-à-tête*. As for knots of plane curve singularities we have for instance that the Lefschetz number of the monodromy vanishes, and hence, a connected sum of tête-à-tête knots is not a tête-à-tête knot. In fact, the trace of the monodromy in first homology equals up to sign the number of terms in such a connected sum, therefore the Lefschetz number will not vanish for connected sums with two or more terms.

Section 1. Pur tête-à-tête retractions to spines and pur tête-à-tête twists.

Let Γ be a finite, connected, metric graph with $e(\Gamma)$ edges and no vertices of valency 1. We assume, that the metric d_Γ on Γ is a path metric. More precisely, each edge is parametrized by a continuous injective map

$$E_e : [0, L_e] \to \Gamma,\ L_e > 0,\ e = 1, \cdots, e(\Gamma),$$

such that

$$d_\Gamma(E_e(t), E_e(s)) = |t - s|,\ t, s \in [0, L_e]$$

holds.

Let Σ be a smooth, connected, compact, oriented surface with non empty boundary $\partial\Sigma$. We say, that a map m of Γ into Σ is a smooth embedding if m is continuous, injective, $m(\Gamma) \cap \partial\Sigma = \emptyset$, the compositions $m \circ E_e,\ e = 1, \cdots, e(\Gamma)$, are smooth embeddings of intervals and moreover, at each vertex v of Γ all outgoing speed vectors of $m \circ E_e,\ v = E_e(0)$ or $v = E_e(L_e)$ are pairwise not proportional by a positive real number.

Given a smooth embedding $m : \Gamma \to \Sigma$, we will denote by abuse of language by the pair (Σ, Γ) the pair $(\Sigma, m(\Gamma))$.

A safe walk along Γ is a continuous injective path $\gamma : [0, \pi] \to \Sigma$ with following the 3 properties:

– $\gamma(t) \in \Gamma,\ t \in [0, \pi]$,

– the speed, measured with the parametrization E_e at $t \in [0, \pi]$ equals ± 1 if $\gamma(t)$ is in the interior of an edge e,

– if the path γ at $t \in (0, \pi)$ reaches the vertex v, the path γ makes the sharpest possible right turn, i.e. the oriented angle at $v = \gamma(t) \in \Sigma$ between the speed vectors $-\dot\gamma(t_-)$ and $\dot\gamma(t_+)$ is among the edges incident with v smallest possible,

It follows, that a safe walk γ is determined by its starting point $\gamma(0)$ and its starting speed vector $\dot\gamma(0)$. Futhermore, from each interior point of an edge start two distinct safe walks.

If we think of the graph as streets with intersections on the surface, we can image a safe walk as a walk staying always at the sidewalk of the street and making only right turns. So, in New York, a safe walk goes around the block by right turns only, and hence, in the same direction as the cars do. In Tokio, a safe walk is even safer, since it is opposite to the direction of the car traffic. In our pictures we draw safe walks by dotted lines near the edges of the graph Γ.

Definition: *Let (Σ, Γ) be the pair of a surface and a regular embedded metric graph. We say that the tête-à-tête property holds for the the pair if*

– *the graph Γ is a regular retract of the surface Σ,*

– *for each point $p \in \Gamma$, p not being a vertex, two distinct safe walks*

$$\gamma'_p, \gamma''_p : [0, \pi] \to \Sigma$$

with $p = \gamma'_p(0) = \gamma''_p(0)$ exist and satisfy moreover $\gamma'_p(\pi) = \gamma''_p(\pi)$.

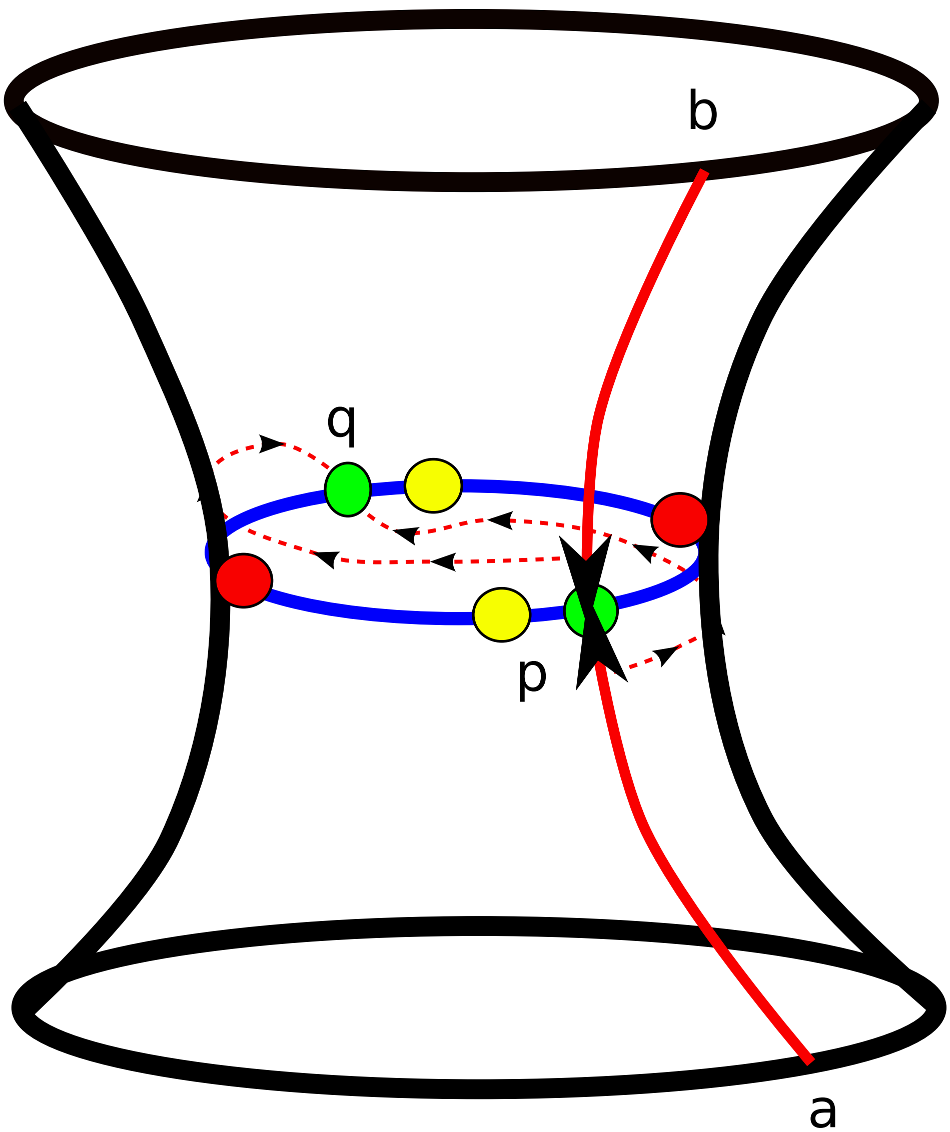

Fig. 1. Blue tête-à-tête graph Γ in $\Sigma_{0,2}$, two retraction intervals from $a, b \in \partial\Sigma$ to $p \in \Gamma$ with the two safe walks from p to $q \in \Gamma$.

Again thinking of the graph as streets, the tête-à-tête property of $\Gamma \subset \Sigma$ means, that two pedestrians being vis-à-vis with respect to the street will be again vis-à-vis after having done simultaneous safe walks of length π.

It follows that the underlying metric graph of a pair (Σ, Γ) with tête-à-tête property is the union of its cycles of length 2π.

We give basic examples of pairs (Σ, Γ) with tête-à-tête property:

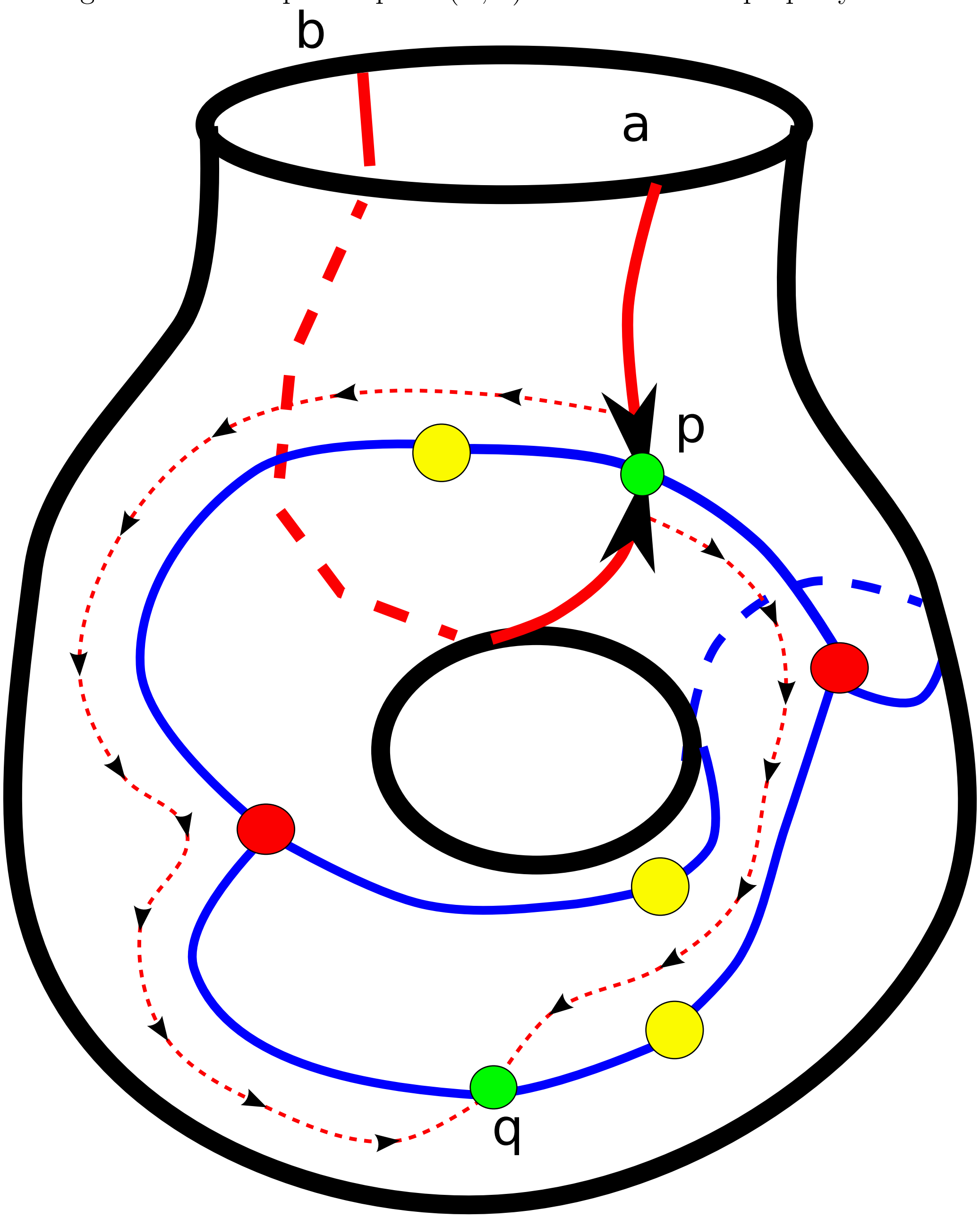

Fig. 2. Tête-à-tête graph $\Gamma := K_{2,3}$ in $\Sigma_{1,1}$. Two retraction intervals from $a, b \in \partial\Sigma$ to $p \in \Gamma$ with the two safe walks from p to $q \in \Gamma$.

— the surface $\Sigma_{0,2}$ is the cylinder $[-1,1] \times S^1$ and the graph Γ is the cycle $\{0\} \times S^1$ subdivided by 4 vertices in 4 edges of equal length $\frac{\pi}{2}$, see Fig 1. Here we think of $\{0\} \times S^1$ as a circle of length 2π or as the bipartite complete graph $K_{2,2}$ with edges of length $\frac{\pi}{2}$.

— the surface $\Sigma_{1,1}$ is of genus 1 with 1 boundary component and the metric graph $\Gamma \subset \Sigma$ is, see Fig. 2, the bipartite complete graph $K_{3,2}$ having edges of length $\frac{\pi}{2}$.

— for $p, q \in \mathbf{N},\ p > 0, q > 0$, the bipartite complete graph $K_{p,q}$ is the spine of a surface $\Sigma_{g,r},\ g = 1/2(p-1)(q-1),\ r = \gcd(p,q)$, such that the tête-à-tête property holds. For instance, let A and B be two parallel lines in the plane and draw p points on A, q points on B. We add pq edges and get a planar projection of the graph $K_{p,q}$. The surface $\Sigma_{g,r}$ is a regular thickening of the graph $K_{p,q}$, such that the given projection of $K_{p,q}$ into the plane extends to an immersion of $\Sigma_{g,r}$ into the plane. Again, we give to all the edges of $K_{p,q}$ length $\frac{\pi}{2}$.

— let P, Q be finite sets, $p = \#P$, $q = \#Q$. The topological join $P * Q$ made with intervals of length $\frac{\pi}{2}$ is again the bipartite metric complete graph $K_{p,q}$. We wish to reconstruct the thickening of the previous example more abstractly. In order to do so, we choose cyclic orderings O_P and O_Q on the sets P and Q. Now, at each vertex of $P * Q$ we have an induced cyclic order on the set of edges incident with that vertex, and hence a well defined (oriented) thickening of the join-graph $P * Q$. It is interesting to observe, that we get by variing the cyclic orderings O_P and O_Q exactly all oriented thickenings of maximal genus of $P * Q$.

Let (Σ, Γ) be a pair of a surface and a graph with tête-à-tête property. Our purpose is to construct for this pair a well defined element T_Γ in the relative mapping class group of the surface Σ. For each edge e of Γ and $p \in e$ such that the point p belongs to the interior of e, we embed relatively a copy $(I_p, \partial I_p)$ of the interval $[-1,1]$ into $(\Sigma, \partial\Sigma)$ such that all copies are pairwise disjoint and such that each copy I_p intersects the graph Γ transversally in its midpoint $0 \in I_p$ in one point which is in the interior of the edge e. We call I_p a dual arc of the edge e. We think of $I_p \subset \Sigma$ as the fiber of a regular retraction of Σ to Γ. Let Γ_p be the union of $\Gamma \cup I_p$. We consider Γ_p also as a metric graph. The graph Γ_p has 2 terminal vertices a, b.

Let $w_a, w_b : [-1, \pi] \to \Gamma_e$ be the only safe walks along Γ_p with $w_a(-1) = a, w_b(-1) = b$. The walk w_a consists of the juxtaposition of the retraction interval I_a from a to $p := w_a(0)$ and one of the two safe walks along Γ that starts at p. The walk w_b is the juxtaposition of a retraction interval I_b and the other safe walk that starts at p. We displace the walks w_a, w_b by a small isotopy to smooth injective path w'_a, w'_b. This isotopy keeps the points $w_a(-1)$, $w_b(-1)$ and $w_a(\pi)$, $w_b(\pi)$ fixed, such that $w'_a(t) \notin \Gamma_e$ for $t \in (-1, \pi)$. The walks w_a, w_b meet each other in $p = w_a(0) = w_b(0) \in e$. Hence, by the tête-à-tête property we have $w_a(\pi) = w_b(\pi)$ and therefore also $w'_a(\pi) = w'_b(\pi)$. Let J_p be the juxtaposition of the paths w'_a and $-w'_b$. We may assume that the path J_p is smooth and intersects Γ transversally. We denote by I_e and J_e the above paths corresponding to the midpoint of the edge e. Moreover, we may assume that the paths J_e, $e = 1, \cdots e(\Gamma)$, are pairwise disjoint. We now claim that there exists up to relative isotopy a unique relative diffeomorphism ϕ_Γ of $(\Sigma, \partial\Sigma)$ with $\phi_\Gamma(I_e) = J_e$. We define the tête-à-tête twist T_Γ as the class of ϕ_Γ.

The following observations prove the above claim. Both the systems of relative arcs I_e and J_e decompose the surface in cells. We say that a cell X of the decomposition by the arcs I_e corresponds to a cell Y of the decomposition by the arcs J_e if X and Y have equal intersections $X \cap \partial\Sigma$ and $Y \cap \partial\Sigma$ with the boundary of Σ. Two corresponding cells have the same number of boundary arcs and of relative arcs. The requirements $\phi_\Gamma(I_e) = J_e$ and ϕ_Γ restricts to the identity on the boundary of Σ define ϕ_Γ on the boundary of the cell X as a homeomorphism to the boundary of the cell Y, provided the cells X and Y correspond to each other. Hence, the class ϕ_Γ can be obtained uniquely by extending to corresponding cells.

The mapping class T_Γ has also the following description. Let Σ_Γ be the space obtained by cutting the surface Σ along Γ. Let $\beta : \Sigma_\Gamma \to \Sigma$ be the tautological map. One can see β as a modification map that is a diffeomorphism above $\Sigma \setminus \Gamma$. The fiber $\beta^{-1}(p)$, $p \in \Gamma$ consist of connected components in the space of continuous maps $c : [0,1] \to \Sigma$ with $c(]0,1]) \subset \Sigma \setminus \Gamma$ and $c(0) = p$, $c(1) \in \partial(\Sigma)$. The connected components of Σ_Γ correspond to components of the boundary of the surface Σ. Each connected component C_i of Σ_Γ is homeomorphic to $\partial_i(\Sigma) \times [0,1]$ where $\partial_i(\Sigma)$, $i = 1, \cdots, r$ runs over the boundary components of Σ. The 1-boundary component of C_i corresponds to $\partial_i(\Sigma)$ while the the 0-boundary component of C_i inherits from the metric graph Γ a metric and a combinatorial structure. Let $T_i : C_i \to C_i$ be a homeomorphism that preserves the projection of C_i to $[0,1]$, that is the identity on the 1-boundary component of C_i, that induces on the 0-boundary component a motion projecting under β to the safe walk on Γ and has only fixed points on the 1-boundary component. Let $T : \cup_{i=1}^r C_i = \Sigma_\Gamma \to \Sigma_\Gamma$. The map T preserves the equivalence relation $\bar\beta$ induced by β on Σ_Γ, hence T induces a self homeomorphism on $\Sigma_\Gamma/\bar\beta = \Sigma$, which is a representative of the class T_Γ.

The above mapping classes $T_\Gamma : \Sigma \to \Sigma$ that are derived from a tête-à-tête graph (Σ, Γ) are very special. We have for instance that length of curves grows linear under interation, more precisely,

Theorem 3. *For each homotopy class c of a closed curve on Σ there exist constants $A, B \in \mathbf{R}$ such that*

$$\text{Length}(T_\Gamma^n(c) \leq A + Bn,\ n \in \mathbf{N}$$

The measurement of length for homotopy classes of closed closed curves on Σ is done by first retracting the curve into the metric graph Γ and by minimizing its path length in its homotopy class in Γ.

Proof. Let c be a closed curve on Σ. First we put c transvers to the graph Γ. By pushing c off to the boundary, we may assume that c consists of a chain of retraction intervals $I_p, p \in \Gamma, p \notin v(\Gamma)$ and intervals lying in the boundary of Σ. The length of such an interval in the boundary does not change by applying T_Γ. For an interval I_p we have $\text{Length}(T_\Gamma^n(I_p)) = 2\pi n$. Hence, $\text{Length}(T_\Gamma^n(c) \leq A + Bn$, where we may put $A = \text{Length}(c)$, $B = 2\pi < c, \Gamma >$ and $< c, \Gamma >$ the minimal unsigned transvers intersection number of c with Γ.

As a corrollary we see that the mapping class T_Γ is of finite type, so its entropy is zero and all eigenvalues of the action of T_Γ on homology are roots of unity. The linear length estimate shows that Jordan blocks are at most of length 1.

For our first basic example $K_{2,2} \subset \Sigma_{0,2}$ we obtain back the classical right Dehn twist. The second example $K_{2,3} \subset \Sigma_{1,1}$ produces a tête-à-tête twist, which is the geometric monodromy of the plane curve singularity $x^3 - y^2$. The twists of the examples $(\Sigma_{g,r}, K_{p,q}),\ p, q \geq 2$, compute the geometric monodromy for the singularities $x^p - y^q$.

The family of Riemann surfaces

$$F_t := \{p \in \mathbf{C}^2 \mid x(p)^3 - y(p)^2 = t,\ ||p|| \leq R,\ t \in \mathbf{C},\ t \neq 0,\ |t| << R\},$$

can be obtained as follows. Let H_t be the interior of the real convex hull in the complex plane $\mathbf{C}$ of $\{s \in \mathbf{C} \mid s^6 = t\}$. The surface $\bar{F}_t := (\mathbf{C} \setminus H_t) \cap D_{R'}$ has two boundary components, one smooth and one with corners. Here we have denoted by $D_{R'}$ the disk of radius $R' >> R$ in $\mathbf{C}$. We subdivide each face of the boundary component with corners by its midpoint obtaining 12 vertices and faces. The Riemann surface F_t is obtained for some choice of R' by gluing orientation reversing two by two the 12 faces. If we enumerate using the complex orientation of H_t the 12 vertices of H_t by $1, 2, \cdots 12$, the gluing is as follows. For i odd glue (cyclicly modulo 12) the face $[i, i+1]$ to $[i+6, i+5]$ and for i even glue the face $[i, i+1]$ to $[i+7, i+6]$. We denote the gluing by N_t and write: $F_t = \bar{F}_t / N_t$. If t runs over a circle $|t| = r, 0 < r << R$, the hexagon H_t rotates by $\frac{2\pi}{6}$, hence the gluing scheme N_t is preserved. So, we obtain a monodromy diffeomorphism $\phi : F_t \to F_t$.

Note that the gluing N_t converges in an appropriate microlocal topology, or perhaps arc space topology, for $t \to 0$ to the normalization of $F_0 = \bar{F}_0 / N_0, \bar{F}_0 := \mathbf{C}$. Conversely, the surface $\bar{F}_t$ can be obtained from $(\Sigma_{1,1}, K_{2,3})$ by cutting the surface $\Sigma_{1,1}$ along the graph $K_{2,3}$.

A similar description holds also for the singularities $x^p - y^q$: replace H_t by the convex hull of $\{s \in \mathbf{C} \mid s^{pq} = t\}$, which is a polygon with pq faces. We get $2pq$ vertices after midpoint subdivision. The gluing N_t is different. For the singularity E_8, given by $x^5 - y^3$, the gluing of the 30 faces of ∂H_t is as follows. For i odd glue (cyclicly modulo 30) the face $[i, i+1]$ to $[i+11, i+10]$ and for i even glue (cyclicly) the face $[i, i+1]$ to $[i+20, i+19]$.

In his seminal work on the ramification of integrals depending upon parameters Frédéric Pham has introduced the graphs $K_{p,q}$ as retracts of the local nearby fibers of the singularities $x^p - y^q$. More generally, Pham has constructed for the singularities $x_0^{a_0} + x_1^{a_1} + \cdots + x_n^{a_n}$ a retraction of the local nearby fiber to a simplicial complex of dimension n consisting of $a_0 a_1 . \cdots . a_n$ n-simplices [F].

Section 2. Relative tête-à-tête retracts and graphs.

We prepare material, that will allow us to glue the previous examples. Let Σ

be a connected compact surface with boundary $\partial\Sigma$. The boundary $\partial\Sigma = A \cup B$ is decomposed as a partition of boundary components of the surface Σ. We assume $A \neq \emptyset, B \neq \emptyset$.

Definition. *A relative tête-à-tête graph* (Σ, A, Γ) *in* (Σ, A) *is an embedded metric graph* Γ *in* Σ *with* $A \subset \Gamma$. *Moreover, the following properties hold:*

– *the graph* Γ *is a regular retract of the surface* Σ,

– *for each point* $p \in \Gamma \setminus A$, p *not being a vertex, two distinct safe walks*

$$\gamma'_p, \gamma''_p : [0, \pi] \to \Sigma$$

with $p = \gamma'_p(0) = \gamma''_p(0)$ *exist and satisfy moreover* $\gamma'_p(\pi) = \gamma''_p(\pi)$.

– *for each point* $p \in A$, p *not being a vertex, one safe walk* γ_p *exists and satisfies* $\gamma_p(\pi) \in A$.

We call the subset A the boundary of the relative tête-à-tête graph (S, A, Γ). This boundary carries a self map

$$w : A \to A,\ p \in A \mapsto \gamma_p(\pi) \in A$$

which is an interval exchange map and which we call the *boundary walk* $w : A \to A$.

We now give a family of examples of relative tête-à-tête graphs.

— Consider the previous example $(\Sigma_{g,r}, K_{p,q})$, $g = 1/2(p-1)(q-1)$, $r = (p, q)$. We blow up in the real oriented sense the p vertices of valency q, so we replace such a vertex $v_i, 1 \leq i \leq p$ by a circle A_i and attach the edges of $K_{p,q}$ that are incident with v_i to the circle in the cyclic order given by the embedding of $K_{p,q}$ in $\Sigma_{g,r}$. We get a surface $\Sigma_{g,r+p}$ and its boundary is partitioned in $A := \cup A_i$ and $B = \partial\Sigma_{g,r}$. The new graph is the union of A with the strict transform of $K_{p,q}$. So the new graph is in fact the total transform $K'_{p,q}$. We think of this graph as a metric graph. The metric will be such that all edges have a positive length and that the tête-à-tête property remains for all points of $K'_{p,q} \setminus A$. We achieve this by giving the edges of A length 2ϵ, $\epsilon > 0$, ϵ small and by giving the edges of $K'_{p,q} \setminus A$ length $\frac{\pi}{2} - \epsilon$. The boundary walk is an interval exchange map from $w : A \to A$. The boundary walk preserves length. We denote by the triple $(\Sigma_{g,r+p}, A, K'_{p,q})$ this relative tête-à-tête graph together with its boundary walk $w : A \to A$.

— Let (Σ, Γ) be a surface with tête-à-tête graph. Let $C \subset \Gamma$ be an embedded closed interval. Let C_w be the image obtained from C by doing all iterated safe walks starting in C. Let Σ'_C be the surface obtained from Σ by cutting along C and C_w. The surface Σ'_C has two boundary components C' and C'_w that correspond to the cutting intervals and a subset Γ' that corresponds to Γ. Put $\Gamma'_C := \Gamma \cup C' \cup C'_w$ and $A := C' \cup C'_w$. The triple (Σ'_C, A, Γ_C) is a relative tête-à-tête graph.

— Let (Σ, A, Γ) be a surface with relative tête-à-tête graph. Let $C \subset \Gamma$ be an embedded closed interval that intersects A in a finite set. Again by cutting along C

we get a transformed surface Σ_C with subsets Γ', C', C'_w, $A_C := A \cup C' \cup C'_w$ and $\Gamma_C := \Gamma' \cup A$, such the triple $(\Sigma'_C, A_C, \Gamma_C)$ is a relative tête-à-tête graph.

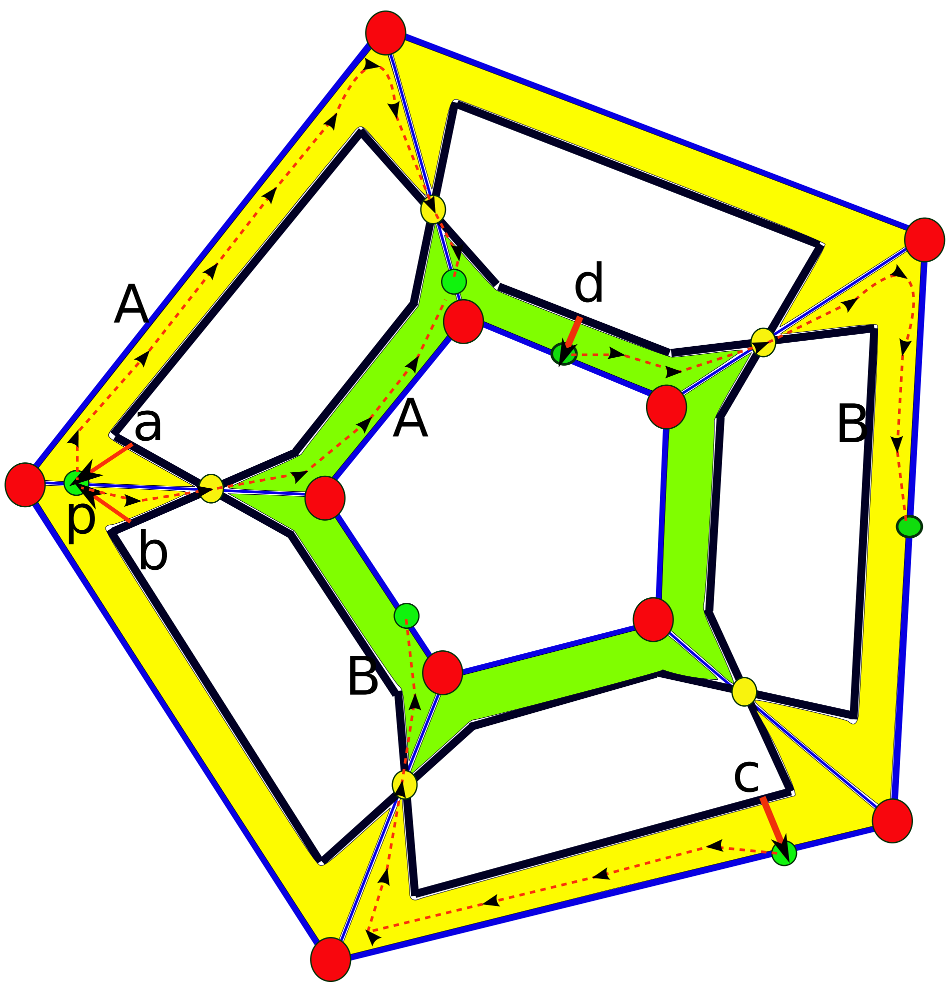

Fig. 3. Blue relative tête-à-tête graph $\Gamma = K'_{2,5}$ in $\Sigma_{2,1+2}$ with two relative blue boundary components A, one black boundary component B; two retraction intervals from $a, b \in B$ with tête-à-tête at $p \in \Gamma \setminus A$, the safe walks from p; two retraction intervals from $c, d \in B$ to points in A with boundary walks in between the components of A. The edges in the relative boundary A have length 2ϵ, the edges in $\Gamma \setminus A$ have length $\frac{\pi}{2} - \epsilon$.

— An iterations of modifications of a tête-à-tête graph (Σ, Γ) or of a relative tête-à-tête graph (Σ, A, Γ) by cutting along closed intervals can be done at once by cutting along a subforest F in Γ provided that $F \cap F_w = \emptyset$ and that F intersects A in a finite set. The first modification by real blowup of a vertex is in fact the modification by cutting along a star shaped tree with center at a vertex.

Let (Σ, A, Γ) be a triple of a surface, a non empty union of boundary components and a graph with relativ tête-à-tête property. Our purpose is to construct for this pair a well defined element $T_{A,\Gamma}$ in the mapping class group of the surface Σ. The mapping class $T_{A,\Gamma}$ will be relativ in the following sense. The mapping $T_{A,\Gamma}$ preserves A and its restriction is the boundary walk $w : A \to A$, moreover, the mapping $T_{A,\Gamma}$ resricts on $\partial\Sigma \setminus A$ to the identity. As before, for each edge e of $\Gamma \setminus A$ and some $p \in e$ in the interior of e, we embed relatively a copy $(I_e, \partial I_e)$ of the interval $[-1, 1]$ into $(\Sigma, \partial\Sigma)$ such that all copies are pairwise disjoint and such that each copy I_e intersects the graph Γ transversally at $0 \in I_e$. Moreover, for each edge $e \subset A$ and some $p \in e$ belonging to the interior of e, we embed relatively a copy $(I_e, \partial I_e)$ of the interval $[0, 1]$ into $(\Sigma, \partial\Sigma)$. All copies are pairwise disjoint. We think of $I_e \subset \Sigma$ as the fiber of a regular retraction of Σ to Γ. Let Γ_e be the union of $\Gamma \cup I_e$. We consider Γ_e also as a metric graph. The graph Γ_e has 2 terminal vertices a, b as before if $e \subset \Gamma \setminus A$ or 1 terminal vertex a if $e \subset A$. As before, we drag the embedded retraction intervals I_a, I_b by safe walks to embedded intervals J_a, J_b. Let J_e be J_a if $e \subset A$ and as before, let J_e be the juxtaposition of J_a, J_b if $e \subset \Gamma \setminus A$.

We now claim that there exists up to relative isotopy a unique relative diffeomorphism $\phi_{A,\Gamma}$ of $(\Sigma, \partial\Sigma)$ with $\phi_{A,\Gamma}(I_e) = J_e$ and having on A as restriction the boundary walk map. We define the tête-à-tête twist $T_{A,\Gamma}$ as the class of $\phi_{A,\Gamma}$.

The following observations prove the above claim. Both the systems of relative arcs I_e and J_e decompose the surface in cells. We say that a cell X of the decomposition by the arcs I_e corresponds to a cell Y of the decomposition by the arcs J_e if X and Y have equal intersections $X \cap \partial\Sigma$ and $Y \cap \partial\Sigma$ with the boundary of Σ. Two corresponding cells have the same number of boundary arcs in A or in $\partial\Sigma \setminus A$ and of relative retraction arcs. The requirements $\phi_{A,\Gamma}(I_e) = J_e$ and ϕ_Γ restricts to the identity on $\partial\Sigma \setminus A$ or to the boundary walk on A define $\phi_{A,\Gamma}$ on the boundary of the cell X as a homeomorphism to the boundary of the cell Y, provided the cells X and Y have the same intersection with $\partial\Sigma \setminus A$.

Section 3. Gluing and closing of relative tête-à-tête graphs to mixed tête-à-tête graphs.

First we describe the procedure of closing. We do it by an example. Consider $(\Sigma_{6,1+2}, A, K'_{2,13})$. We have two relative boundary components A_1 and A_2. In order to close these components, we choose a piece-wise isometric, fix point free, orientation reversing involution $s_1 : A_1 \to A_1$ of the component A_1. The boundary component A_1 will be closed if we identify orbits of the map s_1. In order to get the tête-à-tête property we do the same with the component A_2, but we have to take care: the involution $s_2 : A_2 \to A_2$ should be equivariant via the boundary walk $w : A_1 \to A_2$ to the involution $s_1 : A_1 \to A_1$. Hence, we put

$$p \in A_2 \mapsto s_2(p) := w \circ s_1 \circ w^{-1}(p) \in A_2$$

Unfortunately, as observed by Sebastian Baader, this equivariance property is not always preserved by iteration: the twice iterated boundary walk $w \circ w : A \to A$ maps the component A_1 back to A1. So, we need also for the involution s_1 the property

$$s_1 = w^2 \circ s_1 w^{-2}$$

The boundary components A_1, A_2 are subdivided into 26 intervals of length ϵ and w^2 permutes these intervals by an isometric rotation about 4ϵ, so, the order of w^2 is 13. The possibilities for the involution s_1 are now very restricted, but we still have an example: the orientation reversing interval exchange involution $s_1 : A_1 \to A_1$ that maps the edges of a 26-gon to the opposite edge. We get after closing $(\Sigma_{6,1+2}, A, K'_{2,13})$ by the involutions s_1 and s_2 a pur tête-à-tête graph on $\Sigma_{18,1}$. It turns out that the corresponding tête-à-tête twist on $\Sigma_{18,1}$ is the monodromy diffeomorphism of the iterated torus knot obtained by cabling the torus knot $(2, 13)$ by the torus knot $(2, 13)$.

We wish to modify the above construction in such a way that we can choose more freely the involution $s_1 : A_1 \to A_1$. More concretely, we want for $s_1 : A_1 \to A_1$ an involution that exchanges in an orientation reversing way the opposite edges of a hexagon. If we do so, we would get after closing by the involutions a surface $\Sigma_{8,1}$ with tête-à-tête graph. The corresponding twist would be the geometric monodromy of the singularity $(x^3 - y^2)^2 - 4x^5y - x^7$, see [A′C]. This singularity has two characteristic Puiseux pairs. Its singular fiber is parametrized by $t \in \mathbf{C} \mapsto (t^4, t^6 + t^7) \in \mathbf{C}^2$. The problem is that we would need a hexagon with a symetry of order 13!

In order to overcome this problem we introduce mixed tête-à-tête graphs.

Definition. *A mixed tête-à-tête graph is a triple (Σ, A, Γ) of an oriented compact surface with non empty boundary Σ, an embedded metric graph Γ in Σ and a complete subgraph A in Γ with connected components $A_1, \cdots, A_k$ and numbers $\delta = (\delta_1, \cdots, \delta_k)$. Let $(\Sigma_A,\ \bar{A},\ \Gamma_A)$ the surface with boundary and embedded metric graph that we obtain by cutting Σ along A. We assume that the triple $(\Sigma_A,\ \bar{A},\ \Gamma_A)$ has the properties for being a relative tête-à-tête graph with walks w_A^+, w_A^-. Let $g_A : \Sigma_A \to \Sigma$ the map that glues the boundary components of the surface Σ_A. Let $D_{A,\delta} : \Sigma_A \to \Sigma_A$ be the a boundary Dehn twist of Σ_A that induces in each component C an oriented translation of length $\delta_i, g_A(C) \subset A_i$. Put $\Gamma_{A,\delta} = g_A(D_{A,\delta}(\Gamma_A))$. We consider $w_{A,\delta}^+ = D_{A,\delta} \circ w_A^+$, $w_{A,\delta}^- = D_{A,\delta} \circ w_A^-$. We assume the following mixed tête-à-tête property for each $p \in \Gamma_A$ holds:*

$$g_A(w_{A,\delta}^+(p)) = g_A(w_{A,\delta}^-(p))$$

More briefly, the mixed tête-à-tête property says that both walks starting at $p \in \Gamma_A$ have endpoints $q', q'' \in \Gamma_{A,\delta}$ that glue via g_A to a common point $q = g_A(q') = g_A(q'') \in g_A(\Gamma_{A,\delta}) \subset \Sigma$. As a consequence we have that the mapping class $D_{A,\delta} \circ T_{\Gamma_A} : \Sigma_A \to \Sigma_A$ is compatible with the gluing g_A and induces a mapping class $T_{A,\Gamma}$ on Σ which we call the tête-à-tête twist of the mixed tête-à-tête graph $(\Sigma,\ A\,, \delta)$.

Let $D_{A,\delta}$ be Dehn the boundary Dehn twist of $(\Sigma_{6,1+2}, A, K'_{2,13})$ that induces on the oriented boundary a translation of length δ and that commutes with $\phi_{A,\Gamma} : \Sigma_{6,1+2} \to \Sigma_{6,1+2}$. Let $w_\delta : A \to A$ be the boundary walk on A followed by the restriction of $D_{A,\delta}$ to A. Here, we put $\delta = \frac{1}{6}\epsilon$ and achieve that $w_\delta^2 : A_1 \to A_1$ is of order 6. Indeed, w_δ^2 induces on the component A_1 of length 26ϵ a translation by $2(2\epsilon + \frac{1}{6}\epsilon) = \frac{26\epsilon}{6}$. Let $s_1 : A_1 \to A_1$ be an involution that exchanges in an orientation reversing way the opposite edges of a hexagon, such that s_1 commutes with the restriction of w_δ^2 to A_1.

Let $s_2 : A_2 \to A_2$ be the involution $s_2 = w_\delta \circ s_1 \circ w_\delta^{-1}$. Let $g_A : \Sigma_{6,1+2} \to \Sigma_{8,1}$ the gluing map that closes 2 boundary components by the involutions s_1, s_2. The triple $(g_A(\Sigma_{6,1+2}), g_A(A), g_A(K'_{2,13})$ with the numbers $(\frac{1}{6}, \frac{1}{6})$ is an example of a mixed tête-à-tête graph and the corresponding twist is the monodromy of the singularity $(x^3 - y^2)^2 - 4x^5y - x^7$.

If we make our choices for the involution s_1 generically, the resulting graph Γ on $\Sigma_{8,1}$ will have 43 vertices, 58 edges, 13 vertices of valency 2, 30 vertices of valency 3. The length of the 26 edges that are incident with a vertex of valency 2 is $\frac{\pi}{2} - \epsilon$. The computation of the length of the remaining 32 edges is more difficult. The length of the boundary component A_1 is 26ϵ, hence the total length of the remaining edges is 26ϵ. A generic choice for the hexagon in the metric circle A_1 having a symmetry of order 6 depends on 1 parameter.

We can also do in the same way the closing of $(\Sigma_{6,2+2}, A, K'_{2,12})$. Here we put $\delta = 0$! Hence, we get a pur tête-à-tête graph on $\Sigma_{7,2}$ and the corresponding twist is the monodromy of the symplectic real singularity $(x^3 - y^2)^2 - x^4|x|y : \mathbf{C}^2 \to \mathbf{C}$ and also of a divide link, which is the cabling of the torus knot $(2,3)$ by the torus link $(2,12)$.

Now an example of gluing of two relative tête-à-tête graphs. We glue in a walk equivariant way two copies, say $L =$*left* and $R =$*right* of $(\Sigma_{2,1}, A, K'_{2,5})$. So we glue the exterior boundary A_1^L of length 10ϵ orientation reversing, but isometrically to the exterior boundary A_1^R, see Fig. 3. We have a 1-dimensional family of gluings. Put $\delta = \frac{1}{2}\epsilon$. We glue the interior boundaries A_2^L and A_2^R w_δ-equivariantly. We get a tête-à-tête graph on the surface $\Sigma_{5,2}$. The corresponding twist is the monodromy of the singularity $(x^3 - y^2)(x^2 - y^3)$, see $[A'C]$. We can glue in a special way, such that the 5 vertices on A_1^L match with the 5 vertices on A_1^R. We get a graph Γ on $\Sigma_{5,2}$ with 10 vertices of valency 4 and 10 of valency 2. The 10 edges connecting vertices of valency 4 have length 2ϵ, the 20 edges connecting vertices of valency 2 and 4 have length $\frac{\pi}{2} - \epsilon$. The graph has two embedded cycles of length 10ϵ and 10 embedded cycles of length 2π. Such a cycle of length 10ϵ has 5 edges each of length 2ϵ and a cycle of length 2π has 2 edges of length 2ϵ and 4 edges of length $\frac{\pi}{2} - \epsilon$. We say that two edges of length 2ϵ in this graph are opposite to each other if they belong to a common cycle of length 2π. As a consequence of gluing boundary walk equivariantly we have that each edge of length 2ϵ is opposite to two distinct edges. Now it follows that the action in homology of the tête-à-tête twist is not of finite order.

We can also glue by matching the vertices on A_1^L with midpoints in between vertices on A_1^R. We get a graph Γ on $\Sigma_{5,2}$ with 20 vertices of valency 3 and 10 of valency 2. The 20 edges connecting vertices of valency 3 have length ϵ, the 20 edges connecting vertices of valency 2 and 3 have length $\frac{\pi}{2} - \epsilon$. Putting $\epsilon = \frac{\pi}{4}$ all edges will be of equal length. The notation $(\Sigma_{p,q}, A, K'_{p,q})$ is slightly misleading since the choice of the length ϵ is not made visible. In order to get a metric graph after gluing, the length ϵ should be chosen with care.

We can glue as above also two copies of $(\Sigma_{g,r+2}, A, K'_{2,k}), k \geq 2$ with $g = \frac{k - \gcd(2,k)}{2}$ and $r = \gcd(2,k)$. The cases $k \geq 5$ correspond to the singularities $(x^{k-2} + y^2)(y^{k-2} + x^2)$.

The cas $k = 2$ will be special as was told to me at the Symposium in Higashi-Hiroshima by Mikami Hirasawa: if $k = 2$ each edge of length 2ϵ will be opposite to one edge only and the action in homology will be of finite order. In this case we get a pur tête-à-tête graph $(\Sigma_{1,4}, \Gamma)$ where Γ has 8 vertices and 12 edges, see Fig. 4. One can think first the picture of Fig. 4 as a relative tête-à-tête graph on the square with 4 holes. In order to get an absolut tête-à-tête graph on the surface $\Sigma_{1,4}$ one has to identify on the relativ boundary, here the outer 4-gon, pairs of points by a walk-equivariant fix point free interval echange involution. In our situation the gluing identifies in an orientation reversing way opposite edges of the 4-gon, hence we get the torus minus 4 holes. The graph is a union of 4 simple closed curves on the torus with 4 transversal intersections. The curves come in two pairs of parallel curves. The 4 edges on one pair of parallel curves are subdivided by 4 extra vertices.

The case $k = 4$ is also special, since then we can take $\delta = 0$ and we get a pur tête-à-tête graph $(\Sigma_{7,4}, \Gamma)$.

Remark. Theorem 3 also holds for twist of mixed tête-à-tête graphs. The constant B in the proof has take in account $D_{A,\delta}$. The proof works with $B = 2\pi < c, \Gamma > +2|\delta| < c, A >$, $|\delta| = \text{Maximum}\delta_i$. Actual inear growth appears already for the usual Dehn twist on $S^1 \times [-1, +]$, which is a pur tête-à-tête twist, if one allows for the cycle c a relative cycle. For more complicated examples of pur tête-à-tête graphs actual linear growth can also appear for absolut cycles as for the monodromy of the $(2, 13)$ cabling on the $(2, 13)$ torus knot. The twist of $(\Sigma_{7,4}, \Gamma)$ with the above pur tête-à-tête graph realizes actual linear grows in homology.

Section 4. The laterna property.

Let (Σ, Γ) be a pair of a surface and a graph with tête-à-tête property. The following property is a strengthening of the tête-à-tête property.

– *Laterna property:* from each vertex $v \in \Gamma$ of valency k there exist as a consequence of the tête-à-tête property k safe walks $\gamma_v^i : [0, \pi] \to \Gamma,\ 1 \leq i \leq k$, starting at v and sharing a common endpoint, which is a vertex of valency k. The laterna property asserts, moreover, that the union of their images $\cup_{1 \leq i \leq k} \gamma_v^i([0, \pi])$ is homeomorphic to the bipartite complete graph $K_{2,k}$.

The tête-à-tête graph on $(\Sigma_{1,4}, \Gamma)$ above is again special, it is an example where the laterna property does not hold.

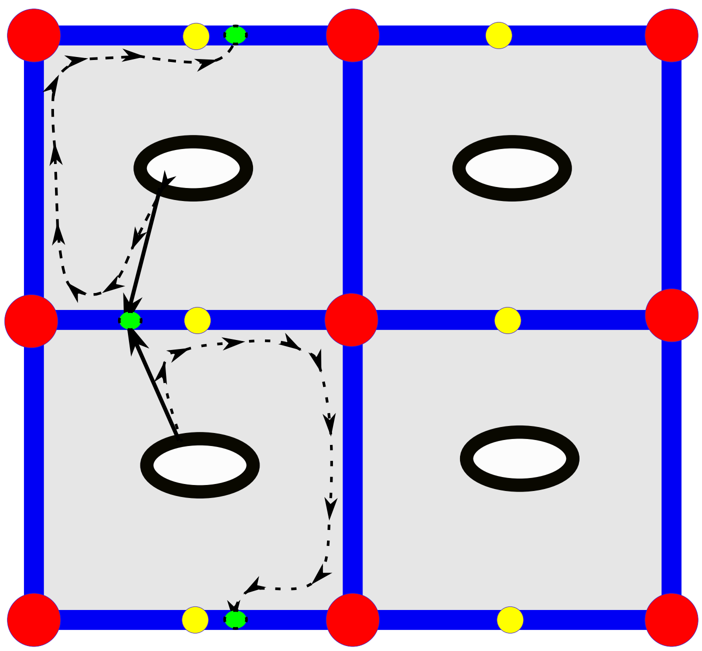

Fig. 4. Blue tête-à-tête graph Γ in the torus minus 4 disks $\Sigma_{1,4}$ with 4 red vertices, 4 yellow vertices and 12 edges. The length of an edge that connects a red and yellow vertex is $\frac{\pi}{2} - \epsilon$ and the length of an edge connecting two red vertices is 2ϵ.

Section 5. **Proofs of Theorem** 1 **and** 2.

– **Th.** 1. First, let $f : \mathbf{C}^2 \to \mathbf{C}$ at 0 define an isolated plane curve singularity having one local branch. The Milnor fiber is constructed in [A'C] as a surface by gluing surfaces that are real blowups in vertices of the tête-à-tête graphs $K_{p,q}$. The pairs (p, q) that appear are characteristic Puiseux pairs of the singularity. The gluings are indeed gluings of relative tête-à-tête graphs. The monodromy as explained in [A'C] permutes the pieces and acts, after iteration giving first return of pieces, in the pieces by the tête-à-tête twists.

For singularities with several branches we use the work of David Eisenbud and Walter Neumann [E-N] and obtain the Milnor fiber as a gluing of relative tête-à-tête graphs, all real blowups of thickenings of the Pham graphs $K_{p,q}$. Again the monodromy permutes pieces and acts by tête-à-tête twists in the pieces.

– **Th.** 2. Let the pair (Σ, Γ) have the tête-à-tête property. Let $T_\Gamma : \Sigma \to \Sigma$ be the corresponding tête-à-tête twist. We realize T_Γ as being the identity outside of a compact subset of $\Sigma \setminus \partial\Sigma$. Let M' be the three manifold $\Sigma \times [0, 1]/R$ where R is the identification of $(p, 0) \in \Sigma \times \{0\}$ to $(T_\Gamma(p), 1) \in \Sigma \times \{1\}$. The boundary of M' is canonically foliated by circles $\{p\} \times S^1$. Let M be the closed three manifold obtained from M' by gluing in solid tori $D^2 \times S^1$ such that meridians are glued to leaves of the boundary foliations. Let $L \subset M$ be the union of the souls of the tori. The pair (M, L) is a fibred link. We claim that M is simply connected, so by the celebrated result of Grigori J. Perelman the manifold M is diffeomorphic to the 3-sphere. It follows that T_Γ is realized as the monodromy diffeomorphism of a classical link. The claim follows from three lemmas.

For an edge e of Γ let $\Lambda_e^+ : S^1 \to \Sigma$ be the free loop in Γ obtained from the juxtaposition of the two safe walks γ_p' and $-\gamma_p''$ starting at the midpoint p of the edge e. The orientation of the loop Λ_e^+ depends on which safe walks starting at p comes first. Let $* \in \Gamma \subset \Sigma \subset M$ be a basepoint. The loop Λ_e^+ defines a conjugacy class K_e^+ in the fundamental group $\pi_1(\Gamma, *) = \pi_1(\Sigma, *)$ and a conjugacy class H_e^+ in $\pi_1(M, *)$. Let $\Lambda_e^- : S^1 \to \Sigma$ be the loop Λ_f^+ where $f = \gamma_e'(\pi) = \gamma_e''(\pi)$. The loops Λ_e^+ and Λ_e^- intersect each other along the edge e and the homological intersection number equals ± 1.

Lemma 1. The union of the conjugacy classes $\cup_{e \in e(\Gamma)} K_e^+$ generates $\pi_1(\Gamma, *) = \pi_1(\Sigma, *)$.

Proof. If the connected graph Γ has only vertices of valency 2, then the graph Γ is homeomorphic to the circle. Hence, for every edge e the image of the loop Λ_e^+ equals Γ and generates $\pi_1(\Gamma, *)$. So, we may assume, that Γ has vertices of valency > 2. Let e be an edge of Γ. A regular neighborhood of the union of Λ_e^+ and Λ_e^- is a subsurface of genus 1 with 1 boundary component. Let $v \in \Gamma$ be a vertex of valency k. Through v run k loops $\Lambda_e^+,\ e \in e(\Gamma),\ v \in e$. By the laterna property the union $G_v := \cup_{v \in e} \Lambda_e^+$ is homeomorphic to the graph $K_{2,k}$. Now we exhaust Γ by taking a maximal sequence of vertices $v_1, v_2, \cdots, v_l$ with

$$v_{i+1} \in \bigcup_{1 \le j \le i} G_{v_j},\ i < k,$$

and

$$\bigcup_{1 \le j \le i} G_{v_j} \setminus G_{v_{i+1}} \neq \emptyset,\ i < k$$

At each step the vertex v_{i+1} will belong to some $G_w \subset \bigcup_{1 \le j \le i} G_{v_j}$ and moreover, we ask for the choice of v_{i+1} that the vertices v_{i+1} and w are the endpoints of an edge in Γ. Let $R_i \subset \Sigma$ be a regular neighborhood of $\cup_{1 \le j \le i} G_{v_j},\ 1 \le i \le k$. We have the following facts: The surfaces R_k and Σ are homotopy equivalent and the conjugacy classes of the loops $\Lambda_e^+,\ e \in e(\Gamma)$, generate $\pi_1(R_k, *)$.

Lemma 2. The union of the conjugacy classes $\cup_{e \in e(\Gamma)} H_e^+$ generates $\pi_1(M, *)$.

Proof. The fundamental group of M' is generated by the fundamental group of the fiber and by a system of meridians. Meridians become trivial in $\pi_1(M, *)$, so the fundamental group of the fiber generates $\pi_1(M, *)$. Lemma 2 now follows from Lemma 1.

Lemma 3. For each edge e the class H_e^+ is the class of the neutral element in $\pi_1(M, *)$.

Proof. For each edge e the loop Λ_e^+ is freely homotopic to the juxtaposition J_e of the arcs I_e and $-T_\Gamma(I_e)$. Let $a, b \in \partial M'$ be the endpoints of I_e. Let M_a, M_b be the oriented meridians through a, b. The loop $I_e * M_a * T_\Gamma(-I_e) * (-M_b)$ bounds in M' an embedded copy of $[0,1] \times I_e$. So the loop J_e is nul homotopic in M, proving that H_e^+ is the conjugacy class of the neutral element in $\pi_1(M, *)$. So the group $\pi_1(M, *)$ is trivial.

Section 6. Concluding remarks.

Remark 1. The real analytic mapping $f : \mathbf{C}^2 \to \mathbf{C}$ given by

$$f(x, y) = (x^3 - y^2)^2 - x^3 \bar{x} y$$

has an isolated singularity at $0 \in \mathbf{C}^2$. This singularity is symplectic in the following sense: locally near the singularity at $0 \in \mathbf{C}^2$ the standard symplectic form

$$\omega := \frac{-1}{2i}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$$

on $\mathbf{C}^2$ restricts at each point p with $\mathrm{Rank}_{\mathbf{R}}(Df)_p = 2$ to a symplectic form on the smooth fiber of f through p. Moreover, the singularity at $0 \in \mathbf{C}^2$ of the map f admits a Milnor type fibration. The monodromy is the twist of the closure, as above, of the graph $(\Sigma_{6,1+2}, A, K'_{2,11})$. The corresponding knot is obtained by cabling the trefoil with the $(2, 11)$ torus knot. This cabling does not satisfy the Puiseux inequalities, so this knot is not the link of an isolated complex plane curve singularity. This knot is however the knot of a divide with 7 crossings, so it still shares many properties with knots of isolated plane curve singularities.

Remark 2. Let (Σ, Γ) be a tête-à-tête graph. From each vertex $v \in \Gamma$ we have constructed a subsurface in Σ, namely the regular neighborhood R of G_v. The valency k_v and the genus $g(R)$ of R are related: $k_v - 2 = g(R)$. So we see that the only tête-à-tête graph in a surface of genus 0 is the graph $K_{2,2}$. Hence the surface $\Sigma_{0,r},\ r > 2$ does not carry a tête-à-tête graph. Let $g(r)$ be the smallest possible genus g such that the surface $\Sigma_{g,r}$ carries a tête-à-tête graph. We guess, that $g(r)$ is the genus of the Milnor fiber of the singularity $x^r + y^r$?

Remark 3. Let $(\Sigma_{g,r}, \Gamma)$ be a tête-à-tête graph. For each oriented edge k of Γ, let $D_k \in H_1(\Sigma, \partial\Sigma, \mathbf{Z})$ be the relative cycle, that is represented by an relatively embedded copy of $[-1, 1]$ dual to the edge k. The cycle D_k is well defined and changes sign by changes the orientation of k. The expression $\delta_k := D_k - T_\Gamma(D_k)$ is an absolute cycle in $H_1(\Sigma, \mathbf{Z})$. The map $D_k \mapsto \delta_k$ is a geometric model for the so called variation map $H_1(\Sigma, \partial\Sigma, \mathbf{Z}) \to H_1(\Sigma, \mathbf{Z})$. We suspect that the cycles δ_k are indeed quadratic vanishing cycles: i.e cycles that vanish at a smooth point of the discriminant in the versal deformation. We enhance Γ by fixing an orientation for each of its edges. The map $k \in e(\Gamma) \mapsto \delta_k \in H_1(\Sigma, \mathbf{Z})$ induces a surjective linear map $\delta : \mathbf{Z}^{e(\Gamma)} \to H_1(\Sigma, \mathbf{Z})$. For each vertex v of Γ, the relative cycle

$$R_v := \sum_{\{k \in e(\Gamma) | v \in k\}} \epsilon_{v,k} D_k$$

where $\epsilon_{v,k} = \pm 1$ is the intersection number of D_k with the oriented edge k_v obtained from k by imposing upon k the orientation "*outgoing from*" v. We have $R_v = 0$ in $H_1(\Sigma, \partial\Sigma, \mathbf{Z})$, hence also

$$\rho_v := \sum_{\{k \in e(\Gamma) | v \in k\}} \epsilon_{v,k} \delta_k = 0$$

in $H_1(\Sigma, \mathbf{Z})$. The fact $R_v = 0$ is very geometric, since the cycle

$$\sum_{\{k \in e(\Gamma) | v \in k\}} \epsilon_{v,k} D_k$$

is the boundary of a relatively embedded disk in $(\Sigma, \partial\Sigma)$. The map $v \in v(\Gamma) \mapsto \rho_v$ induces a map $\kappa : \mathbf{Z}^{v(\Gamma)} \to \mathbf{Z}^{e(\Gamma)}$ with

$$\kappa(v) = \sum_{\{k \in e(\Gamma) | v \in k\}} \epsilon_{v,k} k \in \mathbf{Z}^{e(\Gamma)}$$

Let $\tau : \mathbf{Z} \to \mathbf{Z}^{v(\Gamma)}$ be the linear map with $\tau(1) = \sum_{\{v \in v(\Gamma)\}} \kappa(v)$.

The tête-à-tête twist T_Γ acts on Γ by permutation of the sets $e(\Gamma)$ and $v(\Gamma)$. The action on $e(\Gamma)$ permutes the edges and but does not necessary respect the chosen orientations of the edges. The action of T_Γ on $e(\Gamma)$ and an orientation of the edges leads to a signed permutation matrix. Moreover, the maps τ, κ and δ are T_Γ equivariant, so, since the sequence of maps τ, κ and δ is exact, we get a presentation by signed permutation matrices of semi-simple part of the action of T_Γ upon the homology $H_1(\Sigma, \mathbf{Z})$.

Remark 4. Let $(\Sigma_{g,r}, \Gamma)$ be a tête-à-tête graph describing the monodromy of a plane curve singularity. The alternating product of the characteristic polynomials of the action of T_Γ on $\mathbf{Z}, \mathbf{Z}^{v(\Gamma)}$ and $\mathbf{Z}^{e(\Gamma)}$ is the ζ-function of the monodromy of the singularity. For instance, we get

$$\zeta_{x^p+y^q}(t) = (1 - t^{pq})(1 - t^p)^{-1}(1 - t^q)^{-1}(1 - t)$$

Remark 5. This is work in progress. For isolated singularities of complex hypersurfaces $f : \mathbf{C}^{n+1} \to \mathbf{C}$ we have a construction providing its Milnor fiber with a spine, that consists of lagrangian strata. Again the geometric monodromy is concentrated at the spine. The monodromy diffeomorphism is a generalized tête-à-tête twist. The link of the singularity is decomposed in pieces that are tangent sphere bundles over the langrangian strata. So, we have the possibility of describing combinatorially the contact structure of the link of the singularity.

Remark 6. Let the polynomial $f(x) + g(y)$ have an isolated singularity and assume that we know the lagrangian spines L_f and L_g for the singularities of f and g. The Pham [F] and Sebastiani-Thom [S-T] type results hold: we have that the join $L_f * L_g$ is a lagrangian spine for the singularity of $f + g$ and the monodromy is a join of (generalized) tête-à-tête monodromies. This method leads to new understanding of the monodromy of k-fold suspensions of plane curve singularities $f(x, y) + z^k$. The lagrangian spine is for these singularities a polyhedron build out of $ke(\Gamma_f)$ triangles of curvature $+1$. Along each edge of the tête-à-tête graph Γ_f k twice right angled spherical triangles of curvature +1 meet along their short edge. For each edge the third angle of the k triangles equals the length of the edge. As example after blowing up 2 copies of $K_{2,5}$ with $\epsilon = \pi/6$, and gluing the resulting relativ tête-à-tête graphs, we get a spine with 30 edges of length $\pi/3$ for $(x^3 - y^2)(y^3 - x^2)$, and hence a spine with $30k$ spherical $(\pi/3, \pi/2, \pi/2)$ triangles for $(x^3 - y^2)(y^3 - x^2) + z^k,\ k > 1$. The safe walks needed for the construction of the monodromy project to geodesics in the spherical triangles. We expect a similar picture with general, but not obtuse, spherical triangles for general isolated surface singularities $f(x, y, z)$. In the very special case of Pham-Brieskorn polynomials $x^a + y^b + z^c,\ a, b, c > 1$, the lagrangian spine is build with abc copies of the right angled spherical triangle! See, *Last Minute* below.

Remark 7. The case of plane curves is already interesting for the description of the monodromy as a tête-à-tête twist contains a lot of information and makes hopefully progress in the determination of the adjacency tables of plane curve singularities possible.

Remark 8. The intersection of the cycles δ_k can be computed directly from the graph Γ. So we obtain a Dynkin diagram for the singularity, on which the monodromy acts by permutations. We wonder how this Dynkin diagram relates to the Dynkin diagrams that are obtained by morsification and choices of star shaped systems of paths in the disk?

Last Minute: About hypersurface singularities of higher dimension. In this section we will explain the tête-à-tête monodromy for Pham-Brieskorn type singularities. We give the explanations and notations for the case of surface singularities only. We consider the singularity of

$$f = x^a + y^b + z^c,\ a, b, c > 1,$$

at $0 \in \mathbf{C}^3$. Let A, B, C be cyclicly ordered sets having a, b, c elements respectively. Let $L := A{*}B{*}C$ be the join, which is naturally the realization of a 2-dimensional simplicial complex. A top dimensional face $\sigma_{i,j,k}$ corresponds to a triple $(i, j, k) \in A \times B \times C$. Along the edge (i, j) meet the c faces $\sigma_{i,j,*},\ * \in C$. We will describe in an abstract way a thickening of L proceeding in several steps.

Step 1. Provide each simplex $\sigma_{i,j,k}$ with a riemannian metric that is isometric to the right angled triangle on the euclidian sphere S^2 of radius 1. The radius 1 is choosen such that great circles have length 2π. We identify $\sigma_{i,j,k}$ via the tautological isometry $\phi_{i,j,k} : \sigma_{i,j,k} \to \Delta$ with the right angled triangle Δ that corresponds to the positive octant in the euclidian space $\mathbf{R}^3$. Let U be a regular tubular neighborhood of Δ in the cotangent bundle $T^*(S^2)$ of S^2. Let $U_{i,j,k}$ be a copy of an open regular tubular neighborhood of $\sigma_{i,j,k}$ in $T^*(S^2)$, such that $\phi_{i,j,k}$ extends to an isometry, metrically and also symplectically,

$$\Phi_{i,j,k} : U_{i,j,k} \to U$$

We consider $T^*(S^2)$ as a submanifold in euclidian space $T^*(\mathbf{R}^3) = \mathbf{R}^6$. Let $E \subset T^*(S^2)$ be the equatorial great circle $E := \{(p, L) \in T^*(\mathbf{R}^3) \mid p \in S^2,\ L = 0,\ z(p) = 0\}$ and let $D_E : T^*(S^2) \to \mathbf{R}$ be the distance in $T^*(\mathbf{R}^3)$ of points to E. Finally put $H := \frac{1}{2}D_E^2$.

The Hamiltonian H generates on a regular tubular neighborhood $U_E := \{(p,L) \in T^*(S^2) \mid D_E(p,L) < \epsilon\}$ of E in $T^*(S^2)$ a 2π-periodic flow $\Phi_E(t)$. Having fixed ϵ we can take for

$$U := \{(p,L) \in T^*(S^2) \mid D_\Delta(p,L) < \epsilon\}$$

Step 2. We glue the copies $U_{i,j,k}$ by the following rule. We express this rule for the gluing of $U_{i,j,k}$ and $U_{i,j,k++}$ where $k++$ follows after k in the cyclic order on C. The other gluings are generated by permuting the indices i,j,k and exponents accordingly. In fact, it suffices to give the gluing of $U_{1,1,1}$ and $U_{1,1,2}$. Finally, this gluing is given by the appropriate restiction of

$$\Phi_{1,1,1;1,1,2} := \Phi_{1,1,1}^{-1} \circ \Phi_E(\frac{2}{c}\pi) \circ \Phi_{1,1,2}$$

to $U_E = U_{1,1,1} \cap U_{1,1,2}$.

Step 3. The result of the gluings is a symplectic thickening $\Sigma(A*B*C)$ of the join $A*B*C$, such that $A*B*C$ is a lagrangian spine for it.

Step 4. A safe walk is a continuous curve $\gamma : [0,\pi] \to A*B*C$ such that for each interval $I \subset [0,\pi]$ with $\gamma(I) \subset \sigma_{i,j,k}$ the restriction of γ to I is a geodesic of unit speed in the spherical triangle $\sigma_{i,j,k}$ and more over, for each $t \in]0,\pi[$ with $\gamma(t) \in \sigma_{i,j,k} \cap \sigma_{i',j',k'}$ with $i' = i++$ or $j' = j++$ or $k' = k++$ we have for the left and right speed vectors at t of the curve γ the gluing relation:

$$D(\Phi_{i,j,k;i'j'k'})_{\gamma(t)}(\mathrm{Billard}(\dot\gamma(t_-))) = \dot\gamma(t_+)$$

Here, $\dot\gamma(t_\pm)$ denotes the speed computed as a derivative from the left or right, i.e

$$\dot\gamma(t_\pm) = \lim_{h\to 0, h>0} \frac{\gamma(t \pm h) - \gamma(t)}{\pm h}$$

Moreover, $v \mapsto \mathrm{Billard}(v)$ denotes how on a billard the speed from the left changes to the speed from the right, i.e keeping the tangential component and reversing the normal component of the speed wenn hitting a wall of the billard. One can also say that a safe walk is a broken geodesic: the curve $\gamma : [0,\pi] \to A*B*C$ is a geodesic in the interior of the 2-dimensional faces $\sigma_{i,j,k}$ of the join $A*B*C$ and turns to the next face $\sigma_{i++,j,k}$ or $\sigma_{i,j++,k}$ or $\sigma_{i,j,k++}$ each time it hits the 1-dimensional face of $\sigma_{i,j,k}$ opposite to the vertex i,j or k respectively. If one thinks of the faces $\sigma_{i,j,k}$ as billards, we can say that the billard ball continuous its trajectory on the next billard, each time it hits a wall. The same curve appears if one opens a book, draws a straight line over both pages and closes the book again.

Step 5. The following tête-à-tête property holds: for each $p \in A*B*C$ there exist $q \in A*B*C$ such that all safe walks that start at p end in q. We call q the focal point of p.

Step 6. There exists a unique symplectic mapping class $T_{A*B*C} : \Sigma(A*B*C) \to \Sigma(A*B*C)$ with compact support that is compatible with the tête-à-tête property of safe walks on $\Sigma(A*B*C)$, i.e. T_{A*B*C} maps the relative dual 2-cycle $\delta_{i,j,k}$ to $\sigma_{i,j,k}$ to a relative cycle $T_{A*B*C}(\delta_{i,j,k})$ that intersects $A*B*C$ transversally in the focal point of the intersection point of $\delta_{i,j,k}$ and $\sigma_{i,j,k}$.

Remark 9. The case of morse $x_0^2+x_1^2+\cdots+x_n^2$ singularities is of basic interest. In 1972 Pierre Deligne has explained to me his riemannian model of the relative geometric monodromy T for morse singularities:

$$(p,u)\in T_{\leq 1}S^n \mapsto T(p,u)=(q,v)\in T_{\leq 1}S^n$$

where, if $u\neq 0$, q is the point on the oriented great circle through p with direction u at the oriented distance $(1-||u||^2)\pi$ and v is obtained from u by parallel transport, else, if $u=0$, put $q=-p$, $v=0$. The spine in this case as explained above, consists of 2^{n+1} totally right angled spherical n-simplices and is after gluing the euclidian sphere S^n of radius 1. Its symplectic thickening (with $\epsilon=1$) is the space of covectors $T^*_{\leq 1}S^n$. The symplectic tête-à-tête monodromy of this thickening is after identification of tangent and cotangent space by the metric, in the isotopy class of relative monodromy diffeomorphism of Deligne. Further study from a symplectic viewpoint of this monodromy map is done by Paul Seidel [S].

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