Exercise 1

1. Let $m, n \in \mathbb{Z}$, compute $\frac{\partial}{\partial z}(z^m \bar{z}^n)$ and $\frac{\partial}{\partial \bar{z}}(z^m \bar{z}^n)$.

2. Let U_1 and U_2 be open subets of $\mathbb{C}$. Let $f : U_1 \to \mathbb{C}$ and $g : U_2 \to U_2$ are differentiable, and $g = g_1 + ig_2$. Show that:

$$d(f \circ g) = (\frac{\partial f}{\partial x} \circ g)dg_1 + \frac{\partial f}{\partial y} \circ g)dg_2$$
$$d(f \circ g) = (\frac{\partial f}{\partial z} \circ g)dg + \frac{\partial f}{\partial \bar{z}} \circ g)d\bar{g}$$

.

3. Let U be an open subset of $\mathbb{C}$ and let $\omega \in \Omega^1_k(U)$. Show that for any function f of class C^k on U:

$$d(f\omega) = df \wedge \omega + fd\omega$$

.

4. Let U be a star-shaped open set in $\mathbb{C}$. Show that:
(I) every $\omega = Pdx + Qdx \in \Omega^1_k(U)$ that is closed is exact.
(II) every $\omega = Cdx \wedge dy \in \Omega^2_k(U)$ is exact.
Hint: For (I) for example you can show that the function

$$f(x, y) := x \int_0^1 P(tx, ty)dt + y \int_0^1 Q(tx, ty)dt$$

is a primitive for ω and for (II) look at the 1-form

$$\theta(x, y) := \big(-y \int_0^1 tC(tx, ty)dt\big) dx + \big(x \int_0^1 tC(tx, ty)dt\big) dy$$

5. Le U_1 and U_2 be open subets of $\mathbb{C}$. Let f be a mapping of class C^∞ on U_1 such that $f(U_1) \subset U_2$. Show that for any (differentiable) 0 or 1 form ω on U_2 we have:

$$d(f^*\omega) = f^*(d\omega)$$

6. Let $U_1 =]0, \infty[\times \mathbb{R}$ and $U_2 = \mathbb{C}^*$ be open subsets of $\mathbb{C}$. Let $\rho : U_1 \to U_2$ given by

$\rho_1(r,\theta) = r\cos\theta$ and $\rho_2(r,\theta) = r\sin\theta$. Compute $\rho^*\omega$ for $w = \frac{xdy - ydx}{x^2+y^2}$.

7. Let the path c be the closed polygon $[1-i, 1+i, -1+i, -1-i, 1-i]$. Compute $\int_c \frac{dz}{z}$.

8. Let U be an open subset of $\mathbb{C}$. We denote by $\Omega_c^0(U)$ the set of functions of class C^∞ with compact support in U. Is the operator $\frac{\partial}{\partial \bar{z}} : \Omega_c^0(U) \to \Omega_c^0(U)$ surjective?