Exercise 2

1. Let $r \in \mathbb{R}_+$ and $f : \overline{B(0,r)} \to \mathbb{C}$ be a continous map. Let the path $c : [0, 2\pi] \to \overline{B(0,r)}$ given by $c(t) = re^{it}$ and the family of paths $c_n : [0, 2\pi] \to \overline{B(0,r)}, n \in \mathbb{N}$, given by $c_n(t) = (1 - \frac{1}{n})c(t)$. Show that:
$$\lim_{n\to\infty} \int_{c_n} f(z)dz = \int_c f(z)dz.$$

2. Let U be an open subset of $\mathbb{C}$ and let $\omega \in \Omega_0^1(U)$. Let $c_n : [0,1] \to U$ be a sequence of C^1 paths converging to the path $c : [0,1] \to U$ in C^1-topology. Show that:
$$\lim_{n\to\infty} \int_{c_n} \omega = \int_c \omega.$$

3. Let c be a piecewise C^1 path in $\mathbb{C}$ and $\bar{c}$ its complex conjugate. Let f be a continous function in a neighberhood of $Image(c)$. Show that:
$$\overline{\int_c f(z)dz} = \int_{\bar{c}} \overline{f(\overline{z})}dz.$$
Conclude that if the path $c : [0,1] \to \mathbb{C}$ given by $c(t) = e^{2\pi i t}$ then:
$$\overline{\int_c f(z)dz} = -\int_c \overline{f(z)}\frac{dz}{z^2}.$$

4. Compute the integral $\int_0^{2\pi} \frac{dt}{a^2\cos^2 + b^2 \sin^2}$, for $a, b \in \mathbb{R}_+$, by comparison with the integral $\int_c \frac{dz}{z}$ where the path $c : [0, 2\pi] \to \mathbb{C}^*$ given by $c(t) = a\cos t + ib \sin t$.

5. Let $U = \{x + iy; x, y \in \mathbb{R}_+\}$ and let $f : U \to \mathbb{C}$ be a continous function. Let the paths $c_r : [0, \frac{\pi}{2}] \to U$, $r \in \mathbb{R}_+$, given by $c_r(t) = re^{it}$. We define $M(r) := \sup\{|f(z)|; z \in c_r \setminus \{0\}\}$. Show that if $\lim_{r\to\infty} M(r) = 0$ then:
$$\lim_{r\to\infty} \int_{c_r} f(z)e^{iz}dz = 0.$$

6. Let U be an open subset of $\mathbb{C}$ containing $\overline{B(0,1)}$. Let $H_0(U)$ be the set of holomorphic function f on U such that $f(0) = 0$. Prove that:
(I) If $f \in H_0(U)$ then $\int_0^{2\pi} [f(e^{it})]^4 dt = 0$.
(II) There exists a positive number $M \in \mathbb{R}_+$ such that for every $f \in H_0(U)$:

$$\int_0^{2\pi} [Im(f)(e^{it})]^4 dt \leq M \int_0^{2\pi} [Re(f)(e^{it})]^4 dt.$$