Exercise 3

1. (I) Let U be an open subset of $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be a continuous function. Show that if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on U then f is differentiable on U.
(II) Let $f : \mathbb{C} \to \mathbb{C}$ be defined on real coordinates by $f(x,y) = \frac{2x^2y}{x^4+y^2}$ if $(x,y) \neq (0,0)$ and 0 if $(x,y) = (0,0)$. Show that all directional derivatives of f exist at $(0,0)$ but f is not differentiable at $(0,0)$.

2. Let the path $c : [0,1] \to \mathbb{C}$ given by $c(t) = a + e^{2\pi i t}$ where $a \in \mathbb{C}$. Let U be an open neighborhood of $Image(c)$ and let $\varphi : U \to \mathbb{C}$ be a continuous map. Show that the function $f : D(a,r) \to \mathbb{C}$ defined by $f(z) = \int_c \frac{\varphi(w)}{w-z} dw$ is C^∞.

3. Let $c : [a,b] \to \mathbb{C}$ be continuous, z_0 a complex number not on $Image(c)$. Show that there is a disk D_0 about z_0 disjoint from $Image(c)$ and a continuous function $g : [a,b] \times D_0 \to \mathbb{C}$ such that $c(t) - z = e^{g(t,z)}$ for all $(t,z) \in [a,b] \times D_0$. Moreover, for each $z \in D_0$ and any such function g, the function $t \to g(t,z)$ is C^k if c is a C^k path.

4. Let U be a simply connected open subset of $\mathbb{C}$ and let $f \in H(U)$. Suppose that f is nowhere zero on U. Show that there exists $g \in H(U)$ such that $e^g = f$.

5. Let U be an open subset of $\mathbb{C}$ and let $a, b \in \mathbb{C} \setminus U$. Suppose that a, b lie in the same connected component of $\mathbb{C} \setminus U$. Show that there exists $f \in H(U)$ such that $e^{f(z)} = \frac{z-a}{z-b}$.

6. Let $f \in H(\mathbb{C})$ and $M, N, \alpha \in \mathbb{R}_+$ such that $|f(z)| \leq M + N|z|^\alpha$ for every $z \in \mathbb{C}$. Show that f is a polynomial.

7. Let U be an open subset of $\mathbb{C}$ containing $\overline{D(0,1)}$ and let $f \in H(U)$. Let the path $c : [0, 2\pi] \to \mathbb{C}$ given by $c(t) = e^{it}$. For $|a| \neq 1$ calculate $\int_c \frac{\overline{f(z)}}{z-a} dz$.

8. Let $U = D(0,R)$, $R \in \mathbb{R}_+$, and let $f(z) = \sum_0^\infty a_n z^n$ be a holomorphic function on U. Show that:
(I) Suppose that a_0 is a real number then $f(z) = \frac{1}{2\pi} \int_0^{2\pi} Re(f)(re^{it}) \frac{r+ze^{it}}{r-ze^{it}}$ for every $0 < r < R$.
(II) If $2a_0 = 1$, $R \geq 1$ and $Re(f)(z) \geq 0$ for $|z| < 1$ then $|a_n| \leq 1$ for every $n \in \mathbb{N}$.