Eigenspaces and the Fundamental Theorem of Algebra

Norbert A'Campo

The following well-known theorem will be proved first, and then the so-called Fundamental Theorem of Algebra will be deduced from it.

Theorem 1 *Let V be a $\mathbf{C}$-vectorspace of finite dimension,* $\dim V \geq 2$, *and let $A : V \to V$ be a linear transformation. Then there exists a linear subspace W in V with $A(W) \subset W$ and $0 < \dim W < \dim V$.*

Preliminaries: 1. The angular variation $\theta = \theta(x, y) \in (-\pi/3, \pi/3)$ is defined for $x \in \mathbf{C}, y \in \mathbf{C}$, with $|x - y| < \mathrm{Min}\,\{|x|, |y|\}$ by the equation

$$\frac{y}{x} \cdot \frac{|x|}{|y|} = \cos\theta + i \sin\theta = e^{i\theta}.$$

For $x, y, z \in \mathbf{C}$ with $|x - y| < \mathrm{Min}\,\{|x|, |y|\}$, $|y - z| < \mathrm{Min}\,\{|y|, |z|\}$ and $|x - z| < \mathrm{Min}\,\{|x|, |z|\}$ the angular variations obey the following additive rule: $\theta(x, z) = \theta(x, y) + \theta(y, z)$.

2. For a continuous curve $\gamma : [0, 1] \to \mathbf{C} - \{0\}$ the angular variation along γ is defined to be

$$\theta(\gamma) := \theta_N(\gamma) := \sum_{n=1}^{N} \theta\left(\gamma\left(\frac{n-1}{N}\right), \gamma\left(\frac{n}{N}\right)\right) \in \mathbf{R},$$

where $N \in \mathbf{N}, N \neq 0$, is chosen in such a way that for all $t, t' \in [0, 1]$ with $|t - t'| \leq \frac{1}{N}$ the inequality $|\gamma(t) - \gamma(t')| < \mathrm{Min}\,\{|\gamma(t)|, |\gamma(t')|\}$ holds.

The angular variation $\theta(\gamma)$ does not depend on the actual choice of N, since for two such choices N and N', it follows from the additive rule: $\theta_N(\gamma) = \theta_{NN'}(\gamma) = \theta_{N'}(\gamma)$. The angular variation $\theta(\gamma) = \theta_N(\gamma)$ satisfies:

$$\frac{\gamma(0)}{\gamma(1)} \cdot e^{i\theta(\gamma)} = \prod_{n=1}^{N} \frac{|\gamma(\frac{n-1}{N})|}{|\gamma(\frac{n}{N})|} \in \mathbf{R}_+.$$

3. For a closed continuous curve $\gamma : [0,1] \to \mathbf{C} - \{0\}$ it follows $\frac{1}{2\pi}\theta(\gamma) \in \mathbf{Z}$. To see this, observe: $e^{i\theta(\gamma)} = \frac{\gamma(o)}{\gamma(1)} \cdot e^{i\theta(\gamma)} \in \mathbf{R}_+$.

4. For a constant curve $\gamma : [0,1] \to \mathbf{C} - \{0\}$ we have $\theta(\gamma) = 0$. To see this, compute with $N = 1$.

5. For the curve $\gamma : [0,1] \to \mathbf{C} - \{0\}, \gamma(t) = ae^{2\pi int}, n \in \mathbf{Z}, a \in \mathbf{C}, a \neq 0$, we have $\theta(\gamma) = 2\pi n$. Compute with $N = 6|n| + 1$.

6. For a continuous family $(\gamma_s)_{s\in[0,1]}$ of closed, continuous curves $\gamma_s : [0,1] \to \mathbf{C} - \{0\}$ we have $\theta(\gamma_0) = \theta(\gamma_1)$.

Proof: The function $(t,s) \in [0,1] \times [0,1] \mapsto \gamma_s(t) \in \mathbf{C} - \{0\}$ is uniformly continuous. Choose $N \in \mathbf{N}, N \neq 0$, in such a way that for all $(t,s),(t',s') \in [0,1] \times [0,1]$ with $|t - t'| \leq \frac{1}{N}, |s - s'| \leq \frac{1}{N}$ the inequality $|\gamma_s(t) - \gamma_{s'}(t')| < \text{Min}\,\{|\gamma_s(t)|, |\gamma_{s'}(t')|\}$ holds. The function

$$s \in [0,1] \mapsto \frac{1}{2\pi}\theta(\gamma_s) = \frac{1}{2\pi}\sum_{n=1}^{N} \theta\left(\gamma_s\left(\frac{n-1}{N}\right), \gamma_s\left(\frac{n}{N}\right)\right) \in \mathbf{R}$$

is continuous, with values in $\mathbf{Z}$ by 3, and therefore constant. Hence $\theta(\gamma_0) = \theta(\gamma_1)$. ∎

Proof of the theorem: Choose a basis $f_1, ..., f_d$, $d = \dim(V)$, for V and consider the function

$$\phi : V \to \mathbf{C}, u \in V \mapsto \phi(u) := \det(B_u),$$

where $B_u : V \to V, u \in V$, is the linear transformation given by

$$B_u(f_j) = A^{j-1}(u),\ 1 \leq j \leq d.$$

We have $\phi(\lambda u) = \lambda^{\dim(V)}\phi(u)$, for $\lambda \in \mathbf{C}, u \in V$. For any $w \in W$ with $w \neq 0$ and $\phi(w) = 0$ the subspace $W =: \mathrm{Span}[w, A(w), ..., A^{d-1}(w)]$ satisfies: $A(W) \subset W$ and $0 < \dim(W) < \dim(V)$. Therefore it suffices to prove the existence of a vector $w \in V$ with $w \neq 0$ and $\phi(w) = 0$.

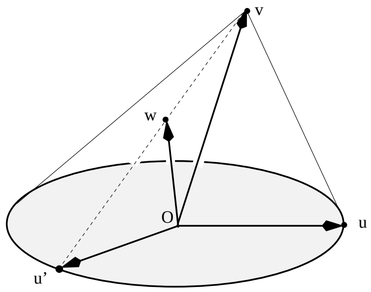

Fig. 1. $a := e^{2\pi it}u,\ b := (1-s)a + sv = (1-s)e^{2\pi it}u + sv$

Choose in V two linear independent vectors u and v, and consider the family of curves $\gamma_s : [0,1] \to \mathbf{C}, s \in [0,1]$ given by:

$$\gamma_s(t) = \phi\left((1-s)\,e^{2\pi it}u + sv\right).$$

We claim: there exists $(t,s) \in [0,1] \times [0,1]$ such that $\gamma_s(t) = 0$, and the vector $w := (1-s)e^{2\pi it}u + sv \in V$ satisfies $w \neq 0$ and $\phi(w) = 0$. We prove the claim by contradiction. Assume $\gamma_s(t) \neq 0$ for all $(t,s) \in [0,1] \times [0,1]$. Observe

$$\gamma_0(t) = e^{2\pi it\dim(V)}\phi(u),\ \ \gamma_1(t) = \phi(v).$$

We obtain: $2\pi\dim(V) = \theta(\gamma_0) = \theta(\gamma_1) = 0$, using the preliminaries 5, 6 and 4, which contradicts the hypothesis $\dim V \geq 2$. ∎

Corollary 1 *Let V be a vectorspace of finite dimension,* $\dim V \geq 1$, *and let $A : V \to V$ a linear transformation. Then there exists a linear subspace L in V with $A(L) \subset L$ and* $\dim L = 1$. *Moreover, every $v \in L$, $v \neq 0$, is an eigenvector, i.e. for some $\lambda \in \mathbf{C}$ the equation $A(v) = \lambda v$ holds. The matrix of A is uppertriangular with respect to some basis of V.*

Proof: Consider a chain of maximal length of linear subspaces in V,$W_0 \subset W_1 \subset ... \subset W_r$ with $A(W_j) \subset W_j$ and $\dim W_{j+1} < \dim W_j$. Then $W_0 =$

$\{0\}, W_r = V$ and A induces on each quotient space W_j/W_{j-1} a linear transformation, say A_j, such that there is no subspace W in W_j/W_{j-1} with $A_j(W) \subset W$ and $0 < \dim(W) < \dim(W_j/W_{j-1})$. So, by the theorem we have $\dim(W_j/W_{j-1}) = 1$. Then $\dim W_j = j$ and the subspace $L := W_1$ has the property of the corollary. A system $e_1, e_2, .., e_r$ such that $e_j \notin W_{j-1}, 1 \leq j \leq r$, is a basis of V for which the matrix of A is uppertriangular. ■

Corollary 2 ("Fundamental Theorem of Algebra") *Let $P(X) \in \mathbf{C}[X]$ be a polynomial of degree ≥ 1. Then their exists $\lambda \in \mathbf{C}$ with $P(\lambda) = 0$.*

Proof: Let $P(X) = X^n + c_{n-1}X^{n-1} + ... + c_0 \in \mathbf{C}[X]$ be a polynomial of degree $n \geq 1$. Consider the linear transformation $A : \mathbf{C}^n \to \mathbf{C}^n$ acting on the standard basis $e_1, e_2, \ldots, e_n$ of $\mathbf{C}^n$ as follows:

$$A(e_j) = e_{j+1},\ 1 \leq j < n, \quad A(e_n) = -\sum_{j=1}^{j=n} c_{j-1}e_j.$$

Clearly $P(A)(e_1) = 0$ and $P(A)(e_j) = P(A)(A^{j-1}(e_1)) = A^{j-1}(P(A)(e_1)) = 0$, $1 < j \leq n$. We conclude that $P(A) = 0$. Thus for an eigenvector $v \in \mathbf{C}^n$, with eigenvalue $\lambda \in \mathbf{C}$, we get $0 = P(A)(v) = P(\lambda)v$, and hence $P(\lambda) = 0$.
■

Remark. The Fundamental Theorem of Algebra was proved by C. Gauss [G]. As is done in the textbooks on Linear Algebra, one can use it also to proof directly the existence of eigenvectors for endomorphism of finite dimensional vector spaces over $\mathbf{C}$. A first appearance of eigenvectors for symetric matrices is in the work of Cauchy on inertials elipsoids (see the textbook of P.M. Cohn [C]). The Jordan Normal Form Theorem is formulated and proved by C. Jordan [J] for general fields k. If any polynomial $\chi(t) \in k[t]$ of positiv degree factors in $k[t]$ in polynomials of degree one, the Normal Form Theorem proofs the existence of invariant flags for endomorphisms of finite dimensional vector spaces over the field k.

[C] P.M. Cohn, Algebra, Wiley, New York, 1982.

[J] Camille Jordan, Traité des substitutions et équations algébriques, Gauthiers-Villars, Paris, 1957.

[G] C. Gauss, .